2.10. Sympy : Symbolic Mathematics in Python

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Objectives

1. Evaluate expressions with arbitrary precision.
2. Perform algebraic manipulations on symbolic expressions.
3. Perform basic calculus tasks (limits, differentiation and integration) with symbolic expressions.
4. Solve polynomial and transcendental equations.
5. Solve some differential equations.

What is SymPy? SymPy is a Python library for symbolic mathematics. It aims become a full featured computer algebra system that can compete directly with commercial alternatives (Mathematica, Maple) while keeping the code as simple as possible in order to be comprehensible and easily extensible. SymPy is written entirely in Python and does not require any external libraries.

Sympy documentation and packages for installation can be found on http://sympy.org/

Chapters contents

2.10.1. First Steps with SymPy

2.10.1.1. Using SymPy as a calculator

SymPy defines three numerical types: Real, Rational and Integer.

The Rational class represents a rational number as a pair of two Integers: the numerator and the denominator, so Rational(1,2) represents 1/2, Rational(5,2) 5/2 and so on:

```
>>> from sympy import *
>>> a = Rational(1,2)

>>> a
1/2

>>> a*2
1
```

SymPy uses mpmath in the background, which makes it possible to perform computations using arbitrary-precision arithmetic. That way, some special constants, like e, pi, oo (Infinity), are treated as symbols and can be evaluated with arbitrary precision:

```
>>> pi**2
pi**2

>>> pi.evalf()
3.14159265358979

>>> (pi+exp(1)).evalf()
5.85987448204884
```

as you see, evalf evaluates the expression to a floating-point number.

There is also a class representing mathematical infinity, called `oo`:

```
>>> oo > 99999
True
>>> oo + 1
oo
```

2.10.1.2. Exercises

1. Calculate $\sqrt{2}$ with 100 decimals.
2. Calculate $1/2 + 1/3$ in rational arithmetic.

2.10.1.3. Symbols

In contrast to other Computer Algebra Systems, in SymPy you have to declare symbolic variables explicitly:

```
>>> from sympy import *
>>> x = Symbol('x')
>>> y = Symbol('y')
```

Then you can manipulate them:

```
>>> x+y+x-y
2*x

>>> (x+y)**2
(x + y)**2
```

Symbols can now be manipulated using some of python operators: +, -, *, ** (arithmetic), &, |, ~ , >>, << (boolean).

2.10.2. Algebraic manipulations

SymPy is capable of performing powerful algebraic manipulations. We'll take a look into some of the most frequently used: expand and simplify.

2.10.2.1. Expand

Use this to expand an algebraic expression. It will try to denest powers and multiplications:

```
In [23]: expand((x+y)**3)
Out[23]: 3*x*y**2 + 3*y*x**2 + x**3 + y**3
```

Further options can be given in form on keywords:

```
In [28]: expand(x+y, complex=True)
```

```
Out[28]: I*im(x) + I*im(y) + re(x) + re(y)

In [30]: expand(cos(x+y), trig=True)
Out[30]: cos(x)*cos(y) - sin(x)*sin(y)
```

2.10.2.2. Simplify

Use simplify if you would like to transform an expression into a simpler form:

```
In [19]: simplify((x+x*y)/x)
Out[19]: 1 + y
```

Simplification is a somewhat vague term, and more precises alternatives to simplify exists: powsimp (simplification of exponents), trigsimp (for trigonometric expressions) , logcombine, radsimp, together.

2.10.2.3. Exercises

1. Calculate the expanded form of $(x + y)^6$.
2. Simplify the trigonometric expression sin(x) / cos(x)

2.10.3. Calculus

2.10.3.1. Limits

Limits are easy to use in SymPy, they follow the syntax limit(function, variable, point), so to compute the limit of f(x) as x -> 0, you would issue limit(f, x, 0):

```
>>> limit(sin(x)/x, x, 0)
1
```

you can also calculate the limit at infinity:

```
>>> limit(x, x, oo)
oo

>>> limit(1/x, x, oo)
0

>>> limit(x**x, x, 0)
```

```
1
```

2.10.3.2. Differentiation

You can differentiate any SymPy expression using `diff(func, var)`. Examples:

```
>>> diff(sin(x), x)
cos(x)
>>> diff(sin(2*x), x)
2*cos(2*x)

>>> diff(tan(x), x)
1 + tan(x)**2
```

You can check, that it is correct by:

```
>>> limit((tan(x+y)-tan(x))/y, y, 0)
1 + tan(x)**2
```

Higher derivatives can be calculated using the `diff(func, var, n)` method:

```
>>> diff(sin(2*x), x, 1)
2*cos(2*x)

>>> diff(sin(2*x), x, 2)
-4*sin(2*x)

>>> diff(sin(2*x), x, 3)
-8*cos(2*x)
```

2.10.3.3. Series expansion

SymPy also knows how to compute the Taylor series of an expression at a point. Use `series(expr, var)`:

```
>>> series(cos(x), x)
1 - x**2/2 + x**4/24 + O(x**6)
>>> series(1/cos(x), x)
1 + x**2/2 + 5*x**4/24 + O(x**6)
```

2.10.3.4. Exercises

1. Calculate $\lim x -> 0, sin(x)/x$
2. Calulate the derivative of log(x) for x.

2.10.3.5. Integration

SymPy has support for indefinite and definite integration of transcendental elementary and special functions via `integrate()` facility, which uses powerful extended Risch-Norman algorithm and some heuristics and pattern matching. You can integrate elementary functions:

```
>>> integrate(6*x**5, x)
x**6
>>> integrate(sin(x), x)
-cos(x)
>>> integrate(log(x), x)
-x + x*log(x)
>>> integrate(2*x + sinh(x), x)
cosh(x) + x**2
```

Also special functions are handled easily:

```
>>> integrate(exp(-x**2)*erf(x), x)
pi**(1/2)*erf(x)**2/4
```

It is possible to compute definite integral:

```
>>> integrate(x**3, (x, -1, 1))
0
>>> integrate(sin(x), (x, 0, pi/2))
1
>>> integrate(cos(x), (x, -pi/2, pi/2))
2
```

Also improper integrals are supported as well:

```
>>> integrate(exp(-x), (x, 0, oo))
1
>>> integrate(exp(-x**2), (x, -oo, oo))
pi**(1/2)
```

2.10.3.6. Exercises

2.10.4. Equation solving

SymPy is able to solve algebraic equations, in one and several variables:

```
In [7]: solve(x**4 - 1, x)
Out[7]: [I, 1, -1, -I]
```

As you can see it takes as first argument an expression that is supposed to be equaled to 0. It is able to solve a large part of polynomial equations, and is also capable of solving multiple equations with respect to multiple variables giving a tuple as second argument:

```
In [8]: solve([x + 5*y - 2, -3*x + 6*y - 15], [x, y])
Out[8]: {y: 1, x: -3}
```

It also has (limited) support for trascendental equations:

```
In [9]: solve(exp(x) + 1, x)
Out[9]: [pi*I]
```

Another alternative in the case of polynomial equations is `factor`. `factor` returns the polynomial factorized into irreducible terms, and is capable of computing the factorization over various domains:

```
In [10]: f = x**4 - 3*x**2 + 1
In [11]: factor(f)
Out[11]: (1 + x - x**2)*(1 - x - x**2)

In [12]: factor(f, modulus=5)
Out[12]: (2 + x)**2*(2 - x)**2
```

SymPy is also able to solve boolean equations, that is, to decide if a certain boolean expression is satisfiable or not. For this, we use the function satisfiable:

```
In [13]: satisfiable(x & y)
Out[13]: {x: True, y: True}
```

This tells us that (x & y) is True whenever x and y are both True. If an expression cannot be true, i.e. no values of its arguments can make the expression True, it will return False:

```
In [14]: satisfiable(x & ~x)
Out[14]: False
```

2.10.4.1. Exercises

1. Solve the system of equations $x + y = 2, 2 \cdot x + y = 0$
2. Are there boolean values `x`, `y` that make `(~x | y) & (~y | x)` true?

2.10.5. Linear Algebra

2.10.5.1. Matrices

Matrices are created as instances from the Matrix class:

```
>>> from sympy import Matrix
>>> Matrix([[1,0], [0,1]])
[1, 0]
[0, 1]
```

unlike a NumPy array, you can also put Symbols in it:

```
>>> x = Symbol('x')
>>> y = Symbol('y')
>>> A = Matrix([[1,x], [y,1]])
>>> A
[1, x]
[y, 1]

>>> A**2
[1 + x*y,     2*x]
[    2*y, 1 + x*y]
```

2.10.5.2. Differential Equations

SymPy is capable of solving (some) Ordinary Differential Equations. sympy.ode.dsolve works like this:

```
In [4]: f(x).diff(x, x) + f(x)
Out[4]:
   2
  d
-----(f(x)) + f(x)
dx dx

In [5]: dsolve(f(x).diff(x, x) + f(x), f(x))
Out[5]: C₁*sin(x) + C₂*cos(x)
```

Keyword arguments can be given to this function in order to help if find the best possible resolution system. For example, if you know that it is a separable equations, you can use keyword hint='separable' to force dsolve to resolve it as a separable equation.

> In [6]: dsolve(sin(x)*cos(f(x)) + cos(x)*sin(f(x))*f(x).diff(x), f(x), hint='separable') Out[6]: -log(1 - sin(f(x))**2)/2 == C1 + log(1 - sin(x)**2)/2

2.10.5.3. Exercises

1. Solve the Bernoulli differential equation x*f(x).diff(x) + f(x) - f(x)**2

⚠ TODO: correct this equation and convert to math directive!

2. Solve the same equation using hint='Bernoulli'. What do you observe ?