Newton's Method to solve simultaneous nonlinear equations

$$\begin{cases} f_1(x_1, x_2, \dots x_N) = 0 \\ \vdots \\ f_N(x_1, x_2, \dots x_N) = 0 \end{cases}$$

There must be the <u>same</u> number of equations as there are variables.

Let me assume that there are two variables and two equations, just to simplify the notation

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

Let us do the Taylor expansion around an estimate x_1, y_1

$$f(x_1, y_1) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_1 \\ y=y_1}} (x - x_1) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_1 \\ y=y_1}} (y - y_1) + \dots$$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_1\\y=y_1}} \qquad \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_1\\y=y_1}}$$

$$g(x_1, y_1) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_1\\y=y_1}} (x-x_1) + \left.\frac{\partial g}{\partial y}\right|_{\substack{x=x_1\\y=y_1}} (y-y_1) + \dots$$

Let us call x^* and y^* the roots that we are looking for. If x_1 and y_1 are close to x^* and y^* it is fair to say that

$$\begin{cases} f(x^*, y^*) = f(x_1, y_1) + \left.\dfrac{\partial f}{\partial x}\right| (x^*-x_1) + \left.\dfrac{\partial f}{\partial y}\right| (y^*-y_1) \\ g(x^*, y^*) = g(x_1, y_1) + \left.\dfrac{\partial g}{\partial x}\right| (x^*-x_1) + \left.\dfrac{\partial g}{\partial y}\right| (y^*-y_1) \end{cases}$$

But by definition $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$

Calling $\Delta x = x_1 - x^*$ and $\Delta y = y_1 - y^*$, we are left with

$$- \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_1\\y=y_1}} \Delta x - \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_1\\y=y_1}} \Delta y + f(x_1, y_1) = 0$$

$$- \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_1\\y=y_1}} \Delta x - \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_1\\y=y_1}} \Delta y + f(x_1, y_1) = 0$$

similar for g

So

$$\begin{cases} \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_1\\y=y_1}}}_{\text{numbers}} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_1\\y=y_1}} \Delta y = f(x_1, y_1) \\ \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_1\\y=y_1}} \Delta x + \left.\frac{\partial g}{\partial y}\right|_{\substack{x=x_1\\y=y_1}} \Delta y = g(x_1, y_1) \end{cases}$$

The variables are Δx and Δy

This is a system of <u>linear</u> <u>equations</u>!

It can also be written in a vector notation

$$\begin{pmatrix} \left.\frac{\partial f}{\partial x}\right|_{\substack{x_1\\y_1}} & \left.\frac{\partial f}{\partial y}\right|_{\substack{x_1\\y_1}} \\ \left.\frac{\partial g}{\partial x}\right|_{\substack{x_1\\y_1}} & \left.\frac{\partial g}{\partial y}\right|_{\substack{x_1\\y_1}} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} f(x_1, y_1) \\ g(x_1, y_1) \end{pmatrix}$$

this is the

Jacobian matrix

In a more general form

$$\boxed{\mathbb{J} \cdot \Delta \mathbb{x} = \mathbb{f}(\mathbb{x})}$$

We solve the system of linear equations using one of the methods we learned (it can also be solve from numpy. linalg)

and use Δx, Δy to find the new estimates x_2 and y_2

We had written: $\Delta x = x_1 - x^*$

$\Delta y = y_1 - y^*$

But we don't have the roots yet, we are trying to get to them, so

$\Delta x = x_1 - x_2$ $\quad$ $x_2 = x_1 - \Delta x$

$$\left.\begin{aligned} \Delta x &= x_1 - x_2 \\ \Delta y &= y_1 - y_2 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x_2 &= x_1 - \Delta x \\ y_2 &= y_1 - \Delta y \end{aligned}}$$

from here, we can restart the procedure