

Newton's Method to solve simultaneous nonlinear equations

$$\left\{ \begin{array}{l} f_1(x_1, x_2, \dots, x_N) = 0 \\ \vdots \\ f_N(x_1, x_2, \dots, x_N) = 0 \end{array} \right. \cdot$$

There must be the same number of equations as there are variables.

Let me assume that there are two variables and two equations, just to simplify the notation

$$\left\{ \begin{array}{l} f(x, y) = 0 \\ g(x, y) = 0 \end{array} \right.$$

Let us do the Taylor expansion around an estimate x_1, y_1

$$\left(f(x_1, y_1) + \frac{\partial f}{\partial x} \Big|_{\substack{x=x_1 \\ y=y_1}} (x-x_1) + \frac{\partial f}{\partial y} \Big|_{\substack{x=x_1 \\ y=y_1}} (y-y_1) + \dots \right)$$

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial x} \Big|_{\substack{x=x_1 \\ y=y_1}} \quad \frac{\partial y}{\partial y} \Big|_{\substack{x=x_1 \\ y=y_1}} \\ \\ g(x_1, y_1) + \frac{\partial g}{\partial x} \Big|_{\substack{x=x_1 \\ y=y_1}} (x-x_1) + \frac{\partial g}{\partial y} \Big|_{\substack{x=x_1 \\ y=y_1}} (y-y_1) + \dots \end{array} \right.$$

Let us call x^* and y^* the roots that we are looking for. If x_1 and y_1 are close to x^* and y^* it is fair to say that

$$\left\{ \begin{array}{l} f(x^*, y^*) = f(x_1, y_1) + \frac{\partial f}{\partial x} \Big|_{\substack{x=x_1 \\ y=y_1}} (x^*-x_1) + \frac{\partial f}{\partial y} \Big|_{\substack{x=x_1 \\ y=y_1}} (y^*-y_1) \\ \\ g(x^*, y^*) = g(x_1, y_1) + \frac{\partial g}{\partial x} \Big|_{\substack{x=x_1 \\ y=y_1}} (x^*-x_1) + \frac{\partial g}{\partial y} \Big|_{\substack{x=x_1 \\ y=y_1}} (y^*-y_1) \end{array} \right.$$

But by definition $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$

Calling $\Delta x = x_1 - x^*$ and $\Delta y = y_1 - y^*$, we are

left with

$$\left[-\frac{\partial f}{\partial x} \Big|_{\substack{x=x_1 \\ y=y_1}} \Delta x - \frac{\partial f}{\partial y} \Big|_{\substack{x=x_1 \\ y=y_1}} \Delta y + f(x_1, y_1) = 0 \right.$$

$$-\left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_1 \\ y=y_1}} \Delta x - \left. \frac{\partial f}{\partial y} \right|_{\substack{x=x_1 \\ y=y_1}} \Delta y + f(x_1, y_1) = 0$$

similar for g

So

numbers

$$\left. \begin{aligned} \left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_1 \\ y=y_1}} \Delta x + \left. \frac{\partial f}{\partial y} \right|_{\substack{x=x_1 \\ y=y_1}} \Delta y &= f(x_1, y_1) \\ \left. \frac{\partial g}{\partial x} \right|_{\substack{x=x_1 \\ y=y_1}} \Delta x + \left. \frac{\partial g}{\partial y} \right|_{\substack{x=x_1 \\ y=y_1}} \Delta y &= g(x_1, y_1) \end{aligned} \right\} \begin{array}{l} \text{The variables} \\ \text{are} \\ \Delta x \text{ and } \Delta y \end{array}$$

This is a system of linear equations!

It can also be written in a vector notation

$$\begin{pmatrix} \left. \frac{\partial f}{\partial x} \right|_{\substack{x_1 \\ y_1}} & \left. \frac{\partial f}{\partial y} \right|_{\substack{x_1 \\ y_1}} \\ \left. \frac{\partial g}{\partial x} \right|_{\substack{x_1 \\ y_1}} & \left. \frac{\partial g}{\partial y} \right|_{\substack{x_1 \\ y_1}} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} f(x_1, y_1) \\ g(x_1, y_1) \end{pmatrix}$$

this is the

Jacobian matrix

In a more general form

$$J \cdot \Delta x = f(x)$$

We solve the system of linear equations using one of the methods we learned

(it can also be solve from `numpy.linalg`)

and use Δx , Δy to find the new estimates x_2 and y_2

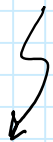
We had written: $\Delta x = x_1 - x^0$

$$\Delta y = y_1 - y^0$$

But we don't have the roots yet, we are trying to get to them, so

$$\Delta x = x_1 - x_2 \quad \left. \vphantom{\Delta x} \right\} \quad \underline{x_2 = x_1 - \Delta x}$$

$$\left. \begin{array}{l} \Delta x = x_1 - x_2 \\ \Delta y = y_1 - y_2 \end{array} \right\} \Rightarrow \boxed{\begin{array}{l} x_2 = x_1 - \Delta x \\ y_2 = y_1 - \Delta y \end{array}}$$



from here, we can
restart the procedure