Tuesday, November 27, 2018 6:34 PM

Euler's formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Some justifications and interpretations

Using derivatives

$$\left.\begin{aligned} f(\theta) &= \cos\theta + i\sin\theta \\ f'(\theta) &= -\sin\theta + i\cos\theta \end{aligned}\right\} \Rightarrow f'(\theta) = i f(\theta)$$

$$\Downarrow$$

$$\boxed{f(\theta) = e^{i\theta}}$$

More details

$$\frac{df}{d\theta} = i f$$

$$\frac{df}{f} = i\, d\theta$$

$$\int \frac{df}{f} = i \int d\theta$$

constant

$$\ln f = i\theta + C$$

$$f = A e^{i\theta}$$

constant

$$\left. f(0) = \cos(0) + i\sin(0) = 1 \right\} \quad A = 1$$

$$\left.\begin{aligned} f(0) &= \cos(0) + i\sin(0) = 1 \\ f(0) &= A e^{i0} = A \end{aligned}\right\} \quad \underline{A = 1}$$

Using Taylor expansion for $\theta \to 0$

$$\Rightarrow \quad e^{i\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 \dots$$

$$\left.\begin{cases} \cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4}\theta^4 \dots \\ \sin\theta = \theta - \frac{\theta^3}{3!} \end{cases}\right\} \Rightarrow \cos\theta + i\sin\theta = e^{i\theta}$$

Interpretation

Suppose we have a complex number

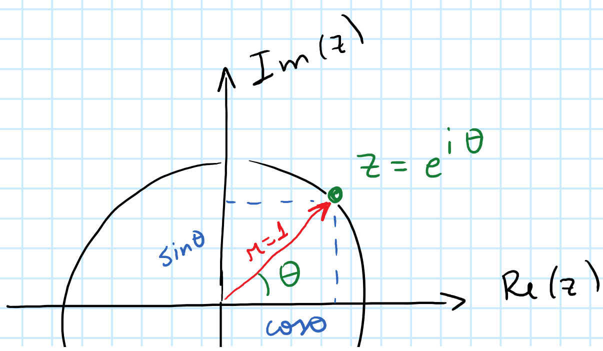
$$z = Re(z) + i\,Im(z)$$

of magnitude 1, that is $|z|^2 = Re(z)^2 + Im(z)^2 = 1$

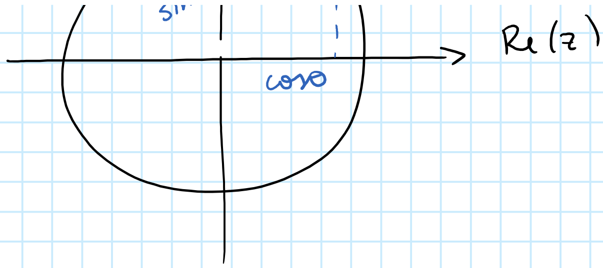
This is satisfied for

$$\left.\begin{aligned} Re(z) &= \cos\theta \\ Im(z) &= \sin\theta \end{aligned}\right\} \quad 0 \le \theta < 2\pi$$

z can be put in the complex plane of radius $= 1$



$$|e^{i\theta}| = 1$$



Due to Euler's formula, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

Analogously, we have hyperbolic functions

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$