Suppose we want to solve the following 4 simultaneous equations for the variables x, y, z, w

$$\begin{cases} 2w + x + 4y + z = -4 \\ 3w + 4x - y - z = 3 \\ w - 4x + y + 5z = 9 \\ 2w - 2x + y + 3z = 7 \end{cases}$$

This can be written in matrix form

$$A x = v$$

where

$$A = \begin{pmatrix} 2 & 1 & 4 & 1 \\ 3 & 4 & -1 & -1 \\ 1 & -4 & 1 & 5 \\ 2 & -2 & 1 & 3 \end{pmatrix} \qquad x = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \qquad v = \begin{pmatrix} -4 \\ 3 \\ 9 \\ 7 \end{pmatrix}$$

→ Gaussian elimination

STEPS → remember that
- we can multiply any equation by a constant
- we can take any linear combination of 2 equations

① Divide the 1st row (A and v) by the top-left element of A.

Here it is ②, so

$$\frac{2}{2} \quad \frac{1}{2} \quad \frac{4}{2} \quad \frac{1}{2} \quad \text{and} \quad \frac{-4}{2}$$

gives

became 1

$$\begin{pmatrix} 1 & \underline{0.5} & \underline{2} & \underline{0.5} \\ 3 & 4 & -1 & -1 \\ 1 & -4 & 1 & 5 \\ 2 & -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \underline{-2} \\ 3 \\ 9 \\ 7 \end{pmatrix}$$

(2) Substitute the 2nd row with

2nd row $-$ (3) $*$ (1st row)

first element in the 2nd row

that is,

$$\underbrace{\begin{pmatrix} 3 & 4 & -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0.5 & 2 & 0.5 \end{pmatrix}}_{\begin{pmatrix} 0 & 2.5 & -7 & -2.5 \end{pmatrix}} \quad \text{and} \quad \underbrace{3 - 3(-2)}_{9}$$

which gives

became zero

$$\begin{pmatrix} 1 & 0.5 & 2 & 0.5 \\ 0 & 2.5 & -7 & -2.5 \\ 1 & -4 & 1 & 5 \\ 2 & -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ 9 \\ 7 \end{pmatrix}$$

(3) Do the same trick between the 3rd and first row,

so that the first element of the third row becomes zero.

Do the same trick between the 4th and 1st row, so that the first element of the fourt row becomes zero

$$
\begin{pmatrix} 1 & 0.5 & 2 & 0.5 \\ 0 & 2.5 & -7 & -2.5 \\ 0 & -4.5 & 1 & 4.5 \\ 0 & -3 & -3 & 2 \end{pmatrix}
\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} =
\begin{pmatrix} -2 \\ 9 \\ 11 \\ 11 \end{pmatrix}
$$

which means

k = 1
2
3 (one before Ntot)

Python code:

```
n=0
div= A[n,n]
A[n,n]= A[n,n]/div
for k in range(n+1, Ntot)
    mult= A[n,k]
    A[k,:]= A[k,:] - mult * A[n,:]
```

(4) Now we move to the 2nd row and perform a similar sequence of operations, so that we will get

$$
\begin{pmatrix} 1 & 0.5 & & \\ 0 & 1 & \cdots & \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}
$$

Python code: same as above but for n = 1

The steps are: •) divide the 2nd row by its 2nd element (here it is 2.5)

•) 3rd row − (−4.5)∗(2nd row)

•) 4th row − (−3)∗(2nd row)

$$
\begin{pmatrix} 1 & 0.5 & 2 & 0.5 \end{pmatrix} \begin{pmatrix} w \end{pmatrix} \begin{pmatrix} -2 \end{pmatrix}
$$

$$\begin{pmatrix} 1 & 0.5 & 2 & 0.5 \\ 0 & 1 & -2.8 & -1 \\ 0 & 0 & -13.6 & 0 \\ 0 & 0 & -11.4 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3.6 \\ 27.2 \\ 21.8 \end{pmatrix}$$

(5) Move to the 3rd row and do the same

- divide by (-13.6)
- 4th row $-$ (-11.4) (3rd row)

Python code
$n = 2$

$$\begin{pmatrix} 1 & 0.5 & 2 & 0.5 \\ 0 & 1 & -2.8 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3.6 \\ -2 \\ -1 \end{pmatrix}$$

Python code
$\boxed{n=3}$
therefore:
for n in range(Ntot):
↳ n = 0, 1, 2, 3

(6) Move to the 4th row

- divide it by -1

$$\begin{pmatrix} 1 & 0.5 & 2 & 0.5 \\ 0 & 1 & -2.8 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3.6 \\ -2 \\ 1 \end{pmatrix}$$

Now we can simply use

⇛ BACKSUBSTITUTION

starting from

$$\begin{cases} z = 1 \\ y = -2 \\ x - 2.8y - z = 3.6 \Rightarrow x = -1 \\ w + 0.5x + 2y + 0.5z = -2 \Rightarrow w = 2 \end{cases}$$

<u>NOTE</u>: Since in Python, the elements start from zero, it is useful to keep the following notation in mind when writing the code

$$\boxed{Ax = v}$$

$$\begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

And after the <u>Gaussian</u> <u>Elimination</u>

$$\begin{pmatrix} 1 & a_{01} & a_{02} & a_{03} \\ 0 & 1 & a_{12} & a_{13} \\ 0 & 0 & 1 & a_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

For the code, consider

$$\begin{cases} N_{tot} = 4 \longrightarrow \text{number of rows (equations)} \\ n \longrightarrow \text{variable for the loop} \\ \qquad \text{for } n \text{ in } \underbrace{\text{range}(N_{tot}-1, -1, -1)}_{\text{this means}} \end{cases}$$

$N_{tot}-1$: $n = 3$

$N_{tot}-2$: $n = 2$

$N_{tot}-3$: $n = 1$

$\Rightarrow$ $N_{tot}-4$: $n = 0$ goes up to one before

$$z = v_3 \Rightarrow x[N_{tot}-1] = \overbrace{v[N_{tot}-1]}^{v_3}$$

$$y + a_{23} z = v_2 \Rightarrow x[N_{tot}-2] = \overbrace{v[N_{tot}-2]}^{v_2} - \overbrace{a[N_{tot}-2, N_{tot}-1]}^{a[2,3]} \, \overbrace{x[N_{tot}-1]}^{z}$$

$$x + a_{12} y + a_{13} z = v_1 \Rightarrow x[N_{tot}-3] = \overbrace{v[N_{tot}-3]}^{v_1} - \overbrace{a[N_{tot}-3, N_{tot}-2]}^{a[1,2]} \, \overbrace{x[N_{tot}-2]}^{y}$$

$$- \overbrace{a[N_{tot}-3, N_{tot}-1]}^{a[1,3]} \, \overbrace{x[N_{tot}-1]}^{z}$$

$$w + a_{01} x + a_{02} y + a_{03} z = v_0 \Rightarrow$$

$$\Rightarrow x[N_{tot}-4] = \overbrace{v[N_{tot}-4]}^{v_0} - \overbrace{a[N_{tot}-4, N_{tot}-3]}^{a[0,1]} \, \overbrace{x[N_{tot}-3]}^{x}$$

$$- \overbrace{a[N_{tot}-4, N_{tot}-2]}^{a[0,2]} \, \overbrace{x[N_{tot}-2]}^{y}$$

$$- \overbrace{a[N_{tot}-4, N_{tot}-1]}^{a[0,3]} \, \overbrace{x[N_{tot}-1]}^{z}$$

```
for n in range (Ntot-1, -1, -1):

    x[n] = v[n]

    for k in range (n+1, Ntot)

        x[n] = x[n] - a[n,k] * x[k]
```

which means
$k = n+1$
$k = n+2$
$\vdots$
$\boxed{k = Ntot - 1}$

PIVOTING

What should we do if we encounter the situation where we would to divide by zero?

Example:

$$\begin{pmatrix} 0 & 1 & 4 & 1 \\ 3 & 4 & -1 & -1 \\ 1 & -4 & 1 & 5 \\ 2 & -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 9 \\ 7 \end{pmatrix}$$

We should swap (pivot) the row with one row below, where the element is farthest from zero (either positive or negative).

In the example above, that means pivoting the 1st and 2nd row ($\underline{\underline{A}}$ and $\underline{\underline{v}}$), and then deal with

$$\begin{pmatrix} \underline{3} & \underline{4} & \underline{-1} & \underline{-1} \\ 0 & 1 & 4 & 1 \\ 1 & -4 & 1 & 5 \\ 2 & -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \underline{3} \\ -4 \\ 9 \\ 7 \end{pmatrix}$$