Jacobian: when we change variables, we distort the area underneath the function.
The Jacobian keeps track of it.

Example:

Suppose we want to solve the integral

$$\int_1^3 x^2\,dx$$

We know how to do it:

$$\left.\frac{x^3}{3}\right|_1^3 = \frac{27-1}{3} = \frac{26}{3}$$

But suppose that instead of solving it as above, we decide to do a transformation of variable

$$y = x^2 \rightarrow x = \sqrt{y}$$

$$dx = \frac{1}{2\sqrt{y}}\,dy$$

this is the Jacobian!

We would still get the same answer for the integral, provided we wouldn't forget the Jacobian

$$\int_1^9 y \frac{dy}{2\sqrt{y}} = \frac{1}{2}\int_1^9 \sqrt{y}\, dy = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3} y^{3/2}\Big|_1^9$$

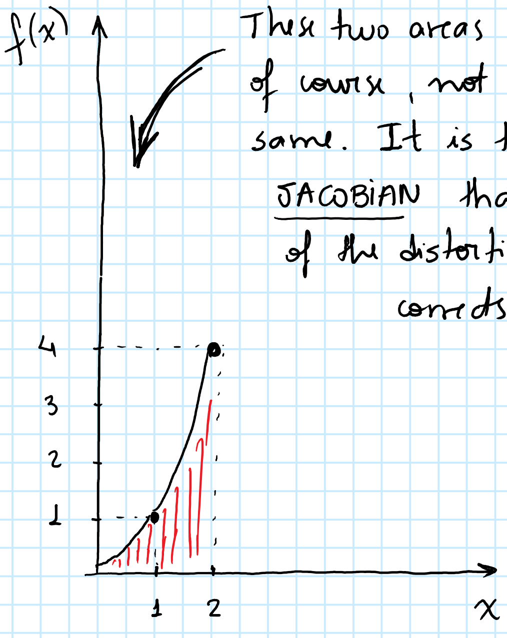
$$= \frac{(\sqrt{9})^3 - 1}{3} = \frac{26}{3}$$

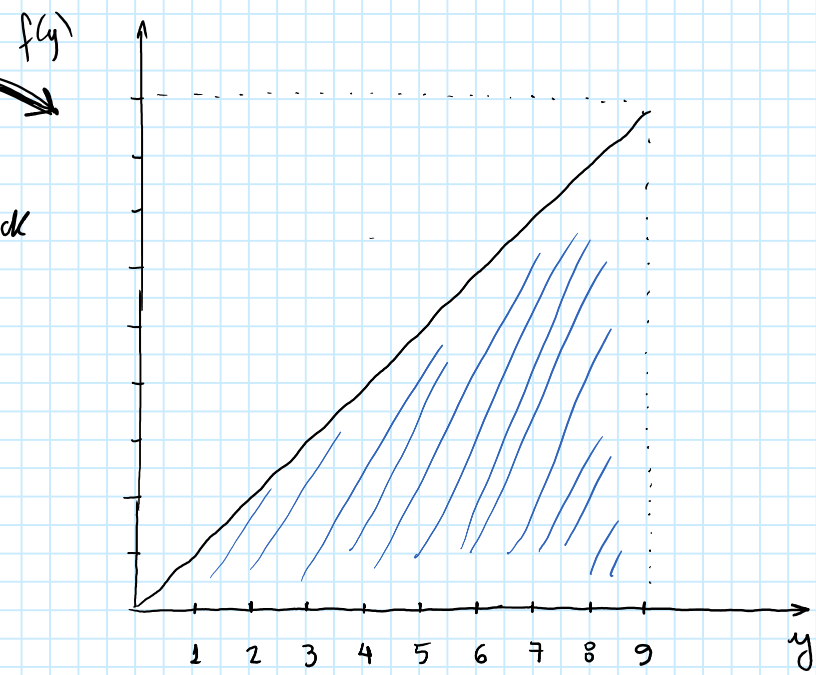
So what is the purpose of the Jacobian?

The original function was x^2 and we had to find the area in red below

For the new function, we have the blue area below

These two areas are, of course, not the same. It is the JACOBIAN that keeps track of the distortion and corrects it.





DOUBLE INTEGRAL

Suppose we have the double integral in x and y

$$\iint dx\, dy$$

and we change the variables to u and v.

The Jacobian is obtained from the determinant

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

So that

$$\iint dx\, dy \longrightarrow \iint J(u,v)\, du\, dv$$

Example: When we solved the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\alpha y^2} dy}$$

we used polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta = r$

Therefore

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy \longrightarrow \int_{0}^{\infty} r\, dr \int_{0}^{2\pi} d\theta$$