Computational Methods in Research

Assignment 5

Exercise 1: Relaxation Method

Consider the equation $x = 1 - e^{-cx}$, where c is a known parameter and x is unknown. This equation arises in a variety of situations, including the physics of contact processes, mathematical models of epidemics, and the theory of random graphs.

1. Write a program to solve this equation for x using the relaxation method for the case $c = 2$. Calculate your solution to an accuracy of at least 10^{-6}.
2. Modify your program to calculate the solution for values of c from 0 to 3 in steps of 0.01 and make a plot of x as a function of c. You should see a clear transition from a regime in which $x = 0$ to a regime of nonzero x. This is another example of a phase transition. In physics this transition is known as the *percolation transition*; in epidemiology it is the *epidemic threshold*.

Exercise 2: Bisection Method

Using the Bisection Method and SCIPY, solve
(i) $x \exp(x) = 1$
(ii) $\cos(x) = x$

Exercise 3: Method of False Position

Using the Method of False Position and SCIPY, solve
(i) $\tan(x) = \frac{1}{1+x^2} \qquad 0 \leq x < \pi/2$
(ii) $\cos(x) = x$
[comparing this item (ii) with the item (ii) above for the bisection method, which method works faster?]

Exercise 4: Newton's Method

Using Newton's Method and SCIPY find the real zero of:
(i) $\arctan(x) = 1 \qquad$ start with $x = 1$
(ii) $\ln(x) = 3 \qquad$ start with $x = 10$

Exercise 5: Wien's displacement constant

Planck's radiation law tells us that the intensity of radiation per unit area and per unit wavelength λ from a black body at temperature T is

$$I(\lambda) = \frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda k_B T} - 1},$$

where h is Planck's constant, c is the speed of light, and k_B is Boltzmann's constant.

One can show by differentiating that the wavelength λ at which the emitted radiation is strongest is the solution of the equation

$$5e^{-hc/\lambda k_B T} + \frac{hc}{\lambda k_B T} - 5 = 0.$$

Substituting $x = hc/\lambda k_B T$, one sees that the wavelength of maximum radiation obeys the *Wien displacement law*:

$$\lambda = \frac{b}{T},$$

where the so-called *Wien displacement constant* is $b = hc/k_B x$, and x is the solution to the nonlinear equation

$$5e^{-x} + x - 5 = 0.$$

1. Write a program to solve this equation to an accuracy of $\epsilon = 10^{-6}$ using the binary search method, and hence find a value for the displacement constant.
2. The displacement law is the basis for the method of *optical pyrometry*, a method for measuring the temperatures of objects by observing the color of the thermal radiation they emit. The method is commonly used to estimate the surface temperatures of astronomical bodies, such as the Sun. The wavelength peak in the Sun's emitted radiation falls at $\lambda = 502\,\text{nm}$. From the equations above and your value of the displacement constant, estimate the surface temperature of the Sun.

Exercise 6: The roots of a polynomial

Consider the sixth-order polynomial

$$P(x) = 924x^6 - 2772x^5 + 3150x^4 - 1680x^3 + 420x^2 - 42x + 1.$$

There is no general formula for the roots of a sixth-order polynomial, but one can find them easily enough using a computer.

1. Make a plot of $P(x)$ from $x = 0$ to $x = 1$ and by inspecting it find rough values for the six roots of the polynomial—the points at which the function is zero.
2. Use numpy to find the roots of the polynomial.

Exercise 7: The roots of a polynomial

Consider the sixth-order polynomial

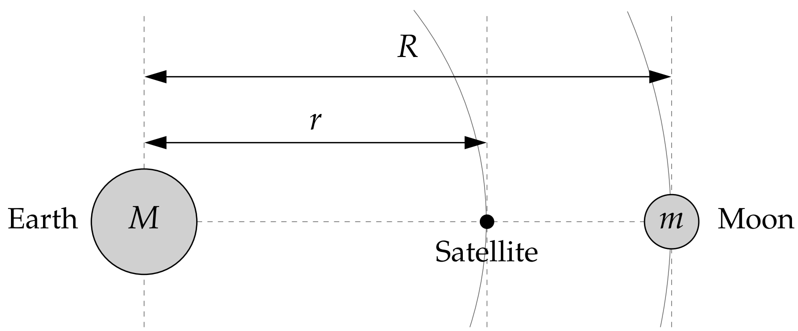
$$P(x) = 924x^6 - 2772x^5 + 3150x^4 - 1680x^3 + 420x^2 - 42x + 1.$$

There is no general formula for the roots of a sixth-order polynomial, but one can find them easily enough using a computer.

1. Write a Python program to solve for the positions of all six roots to at least ten decimal places of accuracy, using Newton's method.

Exercise 8: The Lagrange point

There is a magical point between the Earth and the Moon, called the L_1 Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit. Here's the setup:



Assuming circular orbits, and assuming that the Earth is much more massive than either the Moon or the satellite, one can show that the distance r from the center of the Earth to the L_1 point satisfies

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r,$$

where M and m are the Earth and Moon masses, G is Newton's gravitational constant, and ω is the angular velocity of both the Moon and the satellite.

The equation above is a fifth-order polynomial equation in r (also called a quintic equation). Such equations cannot be solved exactly in closed form, but it's straightforward to solve them numerically. Write a program that uses the secant method to solve for the distance r from the Earth to the L_1 point. Compute a solution accurate to at least four significant figures.

The values of the various parameters are:

$$
\begin{aligned}
G &= 6.674 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}},\\
M &= 5.974 \times 10^{24}\,\mathrm{kg},\\
m &= 7.348 \times 10^{22}\,\mathrm{kg},\\
R &= 3.844 \times 10^{8}\,\mathrm{m},\\
\omega &= 2.662 \times 10^{-6}\,\mathrm{s^{-1}}.
\end{aligned}
$$

You will also need to choose a suitable starting value for r, or two starting values if you use the secant method.

Exercise 9: Newton's Method for Two Variables

Using Newton's Method find the solutions for

$$
\begin{aligned}
f(x,y) &= \exp(3x) + 4y\\
g(x,y) &= 3y^3 - 2\ln(x) + 7.31x^2
\end{aligned}
$$

Use as an initial guess $xo = 1$ and $yo = 2$. Stop when $|f|$ and $|g|$ are smaller than 10^{-5}.

Exercise 10: Maxima and minima

The function $f(x) = x + \sin(5x)$ has three relative maxima and two relative minima in the interval $[0, \pi]$.

(i) Plot the function to get an idea of their locations.

(ii) Find the minima and maxima.