Lorentz Transformation of E and $\vec{p}$

in S

$$E = \frac{m_0}{\sqrt{1-u^2/c^2}}\, c^2$$

$$p_x = \frac{m_0}{\sqrt{1-u^2/c^2}}\, u_x$$

$$p_y = \frac{m_0}{\sqrt{1-u^2/c^2}}\, u_y$$

$$p_z = \frac{m_0}{\sqrt{1-u^2/c^2}}\, u_z$$

$$\boxed{\gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}}$$

$$\boxed{\gamma_{u'} = \frac{1}{\sqrt{1-u'^2/c^2}}}$$

in S'

$$E' = \frac{m_0}{\sqrt{1-u'^2/c^2}}\, c^2$$

$$p'_x = \gamma'_u\, m_0\, u'_x$$

$$p'_y = \gamma_{u'}\, m_0\, u'_y$$

$$p'_z = \gamma_{u'}\, m_0\, u'_z$$

$$1 - \frac{u'^2}{c^2} = 1 - \frac{\left(u_x'^2 + u_y'^2 + u_z'^2\right)}{c^2} = 1 - \frac{\dfrac{(u_x - v)^2}{\left(1 - v/c^2\, u_x\right)^2} + \dfrac{u_y^2}{\gamma^2\left(1 - v/c^2\, u_x\right)^2} + \dfrac{u_z^2}{\gamma^2\left(1 - \frac{v}{c^2} u_x\right)^2}}{c^2} =$$

$$= \frac{c^2\left(1 - \frac{v}{c^2} u_x\right)^2 - \left[u_x^2 - 2u_x v + v^2 - \dfrac{u_x^2}{\gamma^2} + \dfrac{u_x^2}{\gamma^2} + \dfrac{u_y^2}{\gamma^2} + \dfrac{u_z^2}{\gamma^2}\right]}{c^2\left(1 - \frac{v}{c^2} u_x\right)^2} =$$

$$= \frac{c^2 - \cancel{2v u_x} + \overbrace{\frac{v^2}{c^2} u_x^2 - u_x^2}^{-u_x^2/\gamma^2} + \cancel{2v u_x} - v^2 + u_x^2/\gamma^2 - u^2/\gamma^2}{c^2\left(1 - \frac{v}{c^2} u_x\right)^2}$$

$$= \frac{c^2 - \cancel{u_x^2/\gamma^2} - v^2 + \cancel{u_x^2/\gamma^2} - u^2/\gamma^2}{c^2\left(1 - \frac{v}{c^2} u_x\right)^2} = \frac{\overbrace{c^2\left(1 - v^2/c^2\right)}^{c^2/\gamma^2} - u^2/\gamma^2}{c^2\left(1 - v/c^2\, u_x\right)^2} = \frac{\cancel{c^2}\left(1 - u^2/c^2\right)}{\cancel{c^2}\,\gamma^2\left(1 - v/c^2\, u_x\right)^2}$$

Therefore

$$\boxed{\frac{1}{\sqrt{1 - u'^2/c^2}} = \frac{\gamma\left(1 - \frac{v}{c^2} u_x\right)}{\sqrt{1 - u^2/c^2}}}$$

$$E' = \gamma \frac{m_0}{\sqrt{1-u^2/c^2}} \left(1 - \frac{v}{c^2} u_x\right) c^2 = \gamma \left[\gamma_u m_0 c^2 - \gamma_u m_0 u_x v \right] = \gamma \left[E - v p_x \right]$$

$$p_x' = \gamma \frac{m_0}{\sqrt{1-u^2/c^2}} \cancel{\left(1 - \frac{v}{c^2} u_x\right)} \frac{(u_x - v)}{\cancel{\left(1 - \frac{v}{c^2} u_x\right)}} = \gamma \left[\gamma_u m_0 u_x - \gamma_u m_0 c^2 \frac{v}{c^2} \right] = \gamma \left[p_x - \frac{v}{c^2} E \right]$$

$$p_y' = \gamma \frac{m_0}{\sqrt{1-u^2/c^2}} \cancel{\left(1 - \frac{v}{c^2} u_x\right)} \frac{u_y}{\cancel{\left(1 - \frac{v}{c^2} u_x\right)}} = p_y$$

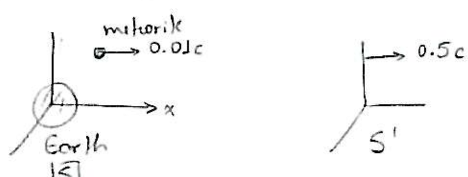
$$p_z' = p_z$$

$$\begin{cases} E' = \gamma (E - v p_x) \\ p_x' = \gamma \left(p_x - \frac{v}{c^2} E\right) \\ p_y' = p_y \\ p_z' = p_z \end{cases} \qquad \begin{cases} E = \gamma (E' + v p_x') \\ p_x = \gamma \left(p_x' + \frac{v}{c^2} E'\right) \\ p_y = p_y' \\ p_z = p_z' \end{cases}$$

Example 2.3

Suppose a micrometeorite of mass 10^{-9} Kg moves past Earth at a speed of $0.01c$. What values will be measured for the energy and momentum of the particle by an observer in a system S' moving relative to Earth at $0.5c$ in the same direction as the micrometeorite?

Let's assume $u_x = 0.01c$ for meteorite $\quad v = 0.5c$ in the x-direction for S'

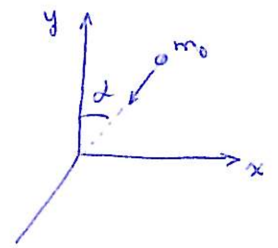


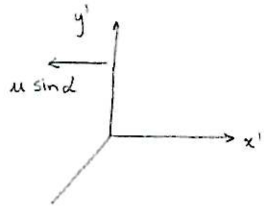
$$\gamma_v = \frac{1}{\sqrt{1-0.5^2}} = 1.1547$$

$$\begin{cases} E = \frac{m_0}{\sqrt{1-u^2/c^2}} c^2 = \frac{10^{-9} (3\times10^8)^2}{\sqrt{1-10^{-4}}} = (1.00005\times10^{-9}\,\text{Kg})\,c^2 \\ p_x = \frac{m_0}{\sqrt{1-u^2/c^2}} u_x = (1.00005\times10^{-11}\,\text{Kg})\,c \end{cases} \qquad \begin{cases} E' = \gamma_v (E - v p_x) = 1.14898\times10^{-9}\,\text{Kg}\,c^2 \\ p_x' = \gamma_v \left(p_x - \frac{v}{c^2} E\right) = -56.6\times10^{-11}\,\text{Kg}\,c \end{cases}$$

Example 2.4 Suppose that a particle with mass m_0 and energy E is moving toward the origin of a system S such that its velocity $\vec{u}$ makes an angle α with the y-axis, as shown in the figure. Using the Lorentz transformation for energy and momentum, determine the energy E' of the particle measured by an observer in S', which moves relative to S so that the particle moves along the y'-axis

S





$\boxed{v = -u \sin\alpha}$

$\Downarrow$

in S' the particle has only the vertical motion along y'

S' sees S moving in the (+)x

$$E' = \gamma_v \left(E + u \sin\alpha \, p_x \right)$$

$$p_x = \frac{m_0}{\sqrt{1 - u^2/c^2}} (-u \sin\alpha) \qquad E = \frac{m_0}{\sqrt{1-u^2/c^2}} c^2$$

$$E' = \frac{1}{\sqrt{1 - \frac{u^2 \sin^2\alpha}{c^2}}} \left(E - \frac{u^2 \sin^2\alpha \, m_0}{\sqrt{1 - u^2/c^2}} \right) = \frac{1}{\sqrt{1 - \frac{u^2 \sin^2\alpha}{c^2}}} \left(E - E \frac{u^2}{c^2} \sin^2\alpha \right) = \frac{E \left(1 - u^2/c^2 \sin^2\alpha\right)}{\sqrt{1 - u^2/c^2 \sin^2\alpha}}$$

$$\boxed{E' = E \sqrt{1 - u^2/c^2 \sin^2\alpha}}$$

when $u \to c \Rightarrow E' = E \cos\alpha$