Relativistic Energy

A particle is initially at rest in x_i with mass m_0

F moves it to x_f

work

$$W = \int_{x_i}^{x_f} F\,dx \qquad F = \frac{dp}{dt} = \frac{d(m(u)\,u)}{dt} = u\frac{dm(u)}{dt} + m(u)\frac{du}{dt}$$

$$W = \int_{x_i}^{x_f}\left(u\frac{dm(u)}{dt} + m(u)\frac{du}{dt}\right)dx$$

$$m(u) = \frac{m_0}{\sqrt{1-u^2/c^2}} \Rightarrow m^2(u)\left(1-u^2/c^2\right) = m_0^2$$

Let us call

the relativistic mass $\overline{\underline{m(u) = m}}$

$$m^2(1-u^2/c^2) = m_0^2$$

$$m^2c^2 - m^2u^2 = m_0^2c^2$$

d/dt

$$\hookrightarrow c^2\frac{dm^2}{dt} - \frac{d(m^2u^2)}{dt} = 0$$

$$\cancel{2}c^2\cancel{m}\frac{dm}{dt} - \cancel{2}\cancel{m}u^2\frac{dm}{dt} - \cancel{2}um^{\cancel{2}}\frac{du}{dt} = 0$$

$$u^2\frac{dm}{dt} + um\frac{du}{dt} = c^2\frac{dm}{dt} \Rightarrow u\frac{dm}{dt} + m\frac{du}{dt} = \frac{c^2}{u}\frac{dm}{dt} = c^2\frac{dm}{d\cancel{t}}\frac{d\cancel{t}}{dx} = c^2\frac{dm}{dx}$$

$$W = \int_{x_i}^{x_f} c^2\frac{dm}{dx}\,dx = c^2\int_{m_i}^{m_f} dm = c^2(m_f - m_i) = \boxed{\frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2}$$

Classical law of energy conservation : total work = change in Kinetic energy

$$\boxed{K = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}} - \underbrace{m_0 c^2}_{\text{rest energy}}}$$ ← relativistic Kinetic energy

• If $u \ll c \Rightarrow \left(1 - u^2/c^2\right)^{-1/2} \simeq 1 + \left[-\frac{1}{2}\left(1-\frac{u^2}{c^2}\right)^{-3/2}(-1)\right]_{\frac{u^2}{c^2}\to 0} u^2/c^2 + \dots$

$$\simeq 1 + \frac{1}{2}\frac{u^2}{c^2}$$

$$K \simeq \cancel{m_0 c^2} + m_0 c^2 \frac{1}{2}\frac{u^2}{c^2} - \cancel{m_0 c^2} = \boxed{\frac{1}{2} m_0 u^2}$$ ← classical Kinetic energy

Relativistic total energy (definition)

$$E = K + m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}}$$

work done by a net force increases the energy of the system from the rest energy $m_0 c^2$ to $\frac{m_0 c^2}{\sqrt{1-u^2/c^2}}$

$$\boxed{E = m c^2}$$ $\left(m = \frac{m_0}{\sqrt{1-u^2/c^2}}\right)$

$\underbrace{E_0 = m_0 c^2}$

mass ⟷ energy

annih / creation of particles

cons. of energy, momentum, charge ...

→ symmetry Emmy Noether

• For $u \ll c \Rightarrow E = \frac{1}{2} m_0 u^2 + m_0 c^2$

(!) NOT in conflict with classical physics

Zero of energy is arbitrary so

we can add this constant $(m_0 c^2)$ to it

$E = mc^2$ $\qquad$ $E = \frac{m_0}{\sqrt{1-u^2/c^2}} c^2$

It is common to express the masses of particles in energy units

$\underbrace{1.0\ eV}_{\text{amount of energy for}} = \underbrace{1.602 \times 10^{-19}\ C}_{\text{an electron to accelerate}} \times \underbrace{1.0\ V}_{\text{through a potential difference of 1 volt}} = 1.602 \times 10^{-19}\ J$

Example: mass of electron at rest $= \boxed{9.11 \times 10^{-31}\ kg}$

Its rest energy is $E = m_0 c^2 = 9.11 \times 10^{-31}\ kg \cdot c^2 = 8.19 \times 10^{-14}\ J$

$E = \frac{8.19 \times 10^{-14}\ J}{1.602 \times 10^{-19}\ J}\ 1\ eV = 5.11 \times 10^5\ eV \Rightarrow E = 0.511\ MeV$ rest energy of the electron

$\Rightarrow m_0 = \frac{E}{c^2} = \boxed{0.511\ MeV/c^2}$ mass of the electron at rest

Example 2.5

Muon speed relative to Earth = 0.998c. If its rest energy is 105.7 MeV, what will observers on Earth measure for its total energy? What will they measure for its mass?

$E_{rest} = m_0 c^2 = 105.7\ MeV \Rightarrow E = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}} = \frac{105.7}{\sqrt{1-(0.998)^2c^2/c^2}} = \boxed{1670\ MeV}$

$m = \frac{E}{c^2} = \boxed{1670\ MeV/c^2}$

It is often convenient to have an expression for the total relativistic energy in terms of momentum

$$E^2 - m_0^2 c^4 =$$

$$m^2 c^4 - m_0^2 c^4 = m_0^2 c^4 \left(\frac{1}{1 - u^2/c^2} - 1 \right) = \frac{m_0^2 c^4 \; u^2/c^2}{1 - u^2/c^2}$$

$$= \frac{m_0^2 c^2 u^2}{1 - u^2/c^2} = c^2 p^2$$

$$\Rightarrow m^2 c^4 = c^2 p^2 + m_0^2 c^4$$

$$\boxed{E^2 = c^2 p^2 + m_0^2 c^4}$$

Example 2-9 A particular object is observed to move through the lab at high speed. Its total energy $E = 4.5 \times 10^{17}$ J and $p_x = 3.8 \times 10^8$ Kg m/s, $p_y = 3.0 \times 10^8$ Kg m/s, $p_z = 3.0 \times 10^8$ Kg m/s. What is the object's rest mass?

$$m_0^2 = \frac{E^2}{c^4} - \frac{p^2}{c^2} = \left(\frac{4.5 \times 10^{17}}{c^2}\right)^2 - \left[\left(\frac{3.8 \times 10^8}{c}\right)^2 + \left(\frac{3.0 \times 10^8}{c}\right)^2 + \left(\frac{3.0 \times 10^8}{c}\right)^2\right]$$

$$\boxed{m_0 = 4.6 \text{ Kg}}$$

Example 2-10 The total energy of an e^- produced in a reaction is 2.40 MeV. Find the e's momentum and speed ($m_0 = 9.11 \times 10^{-31}$ Kg, rest energy $= 0.511$ MeV)

- $pc = \sqrt{E^2 - m_0^2 c^4} = \sqrt{(2.40)^2 - (0.511)^2} \Rightarrow \boxed{p = 2.34 \text{ MeV}/c}$

- $E = \dfrac{m_0 c^2}{\sqrt{1 - u^2/c^2}}$ and $p = \dfrac{m_0}{\sqrt{1 - u^2/c^2}} u \Rightarrow E = \dfrac{p}{u} c^2 \Rightarrow$

$$\Rightarrow \boxed{u = \frac{pc}{E} c} \Rightarrow u = \frac{2.34}{2.40} c \Rightarrow \boxed{u = 0.975c}$$

Massless Particles

$m_0 c^2 = 0 \rightarrow$ idea of zero rest mass has no analog in classical physics

$\Downarrow$

$\boxed{E = pc}$ and from $\frac{u}{c} = \frac{pc}{E} \Rightarrow \boxed{u = c}$

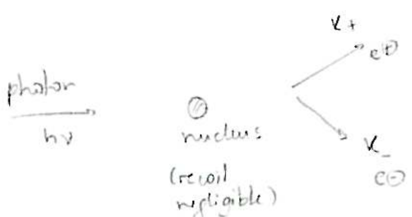
A particle whose mass is zero moves at the speed of light

(photon, gluon, graviton)

Creation and Annihilation of particles

Equivalence of mass and energy $\Rightarrow$

- particles combine with antiparticles and masses of both convert to energy
- mass can be created from energy



$h\nu = K_+ + m_0c^2 + K_- + m_0c^2 = K_+ + K_- + 2m_0c^2$

Example 2-6 (Eisberg) Analysis of a bubble chamber photograph reveals the creation of e^-e^+ pair as photons pass through matter.

$B = 0.20\ Wb/m^2$, $r = 2.5\times10^{-2} m$, { p first then $E, \lambda = ?$ of photon }

$quB = \frac{mu^2}{R} \Rightarrow (p = qBR = (1.6\times10^{-19} C)(0.20 W/m^2)(2.5\times10^{-2} m) = 8\times10^{-22} kg\,m/s$

$E_{\ominus}^2 = p^2c^2 + m_0^2c^4 = \Big(\underbrace{\frac{8\times10^{-22}\times3\times10^8}{1.6\times10^{-19}}}_{MeV}\Big)^2 + (0.511\ MeV)^2 \Rightarrow E_- = 1.6\ MeV$

positron same K $\Rightarrow E_+ = 1.6\ MeV$

$\boxed{E_{photon} = 3.2\ MeV}$

$E = h\nu = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{6.6\times10^{-34} J.s \times 3\times10^8 m/s}{3.2\times10^6 \times 1.6\times10^{-19} J/eV} \Rightarrow \boxed{\lambda = 0.0039 Å}$