Galilean transformation

\[ x' = x - v t \]
\[ t' = t \]

Under Lorentz transformation

\[ a^t = \frac{d\dot{x}}{dt} \]
\[ \ddot{x} = \frac{d^2x}{dt^2} \]
\[ F_x = \frac{\dot{x}^2}{c^2} \]
\[ m_x = \frac{\dot{x}^2}{c^2} \]

F = ma is not invariant

because \( a^t \neq a \)

\[ a^t = \frac{d\dot{x}}{dt} = \frac{d^2x}{dt^2} \left( 1 - \frac{\dot{x}^2}{c^2} \right) \]

\[ \frac{a x}{c^2} = \frac{\dot{x}^2}{c^2} \left( 1 - \frac{\dot{x}^2}{c^2} \right) \]

F = ma would increase the velocity indefinitely. Relativistic mass (inertia) increases with the speed, preventing that v ever becomes > than c.
Relativistic Mass

Lorentz transform = need modification of the equations of mechanics so that they remain invariant under the transformation from one initial frame to another.

\[ \Rightarrow \text{RELATIVISTIC mechanics} \]

Newton's 2nd law in the form \( F = ma \) is (not) relativistically invariant.

but

\[ F = \frac{dp}{dt} \] is relativistically invariant if the relativistic momentum \( \frac{p}{\sqrt{1 - \beta^2}} \) is used.

\[ \Rightarrow \text{mass is a function of } v \]

Thought experiment

From reference frame \( S' \)

\[ \begin{align*}
\text{observer } O_2 & \text{ moves to the left} \\
\text{O}_2 & \text{ with respect to } S
\end{align*} \]

conservation of momentum

elastic collision

From the reference frame of \( O_2 \)

\( \vec{v} \): velocity of \( O_2 \) with respect to \( O_1 \)

From the reference frame of \( O_2 \)

\( \vec{v} \): velocity of \( O_2 \) with respect to \( O_2 \)
From the reference frame of $O_1$ $(S)$

Ball 1 moves along the $y$ axis with velocity $my_1 = u_0$

Ball 2 has an $x$ and $y$ components of its velocity

$$\left\{\begin{array}{l}
mx_2 = mx_2 + \frac{v}{c^2} \frac{my_2}{c^2} \\
mv = \frac{my_2}{c^2} \frac{vy}{c^2}
\end{array}\right. \rightarrow mx_2 = v
$$

Before $my_1 = u_0$

After $my_2 = -u_0$

Collision $my_2 = -\frac{u_0}{c^2}$

Collision $my_2 = \frac{u_0}{c^2}$

球在参考系$O_1$ $(S)$中沿$y$轴运动，速度为$my_1 = u_0$

球2有两个$x$和$y$方向的分速度

$$\left\{\begin{array}{l}
mx_2 = mx_2 + \frac{v}{c^2} \frac{my_2}{c^2} \\
mv = \frac{my_2}{c^2} \frac{vy}{c^2}
\end{array}\right. \rightarrow mx_2 = v
$$

碰撞前$my_1 = u_0$

碰撞后$my_2 = -u_0$

碰撞前$my_2 = -\frac{u_0}{c^2}$

碰撞后$my_2 = \frac{u_0}{c^2}$

为了保证碰撞后在$S$系中动量守恒

$$\begin{align*}
(m(m_0) u_0 - m(u) \frac{m}{m_2} = - m(u_0) u_0 + m(u) \frac{m}{m_2}
\end{align*}$$

$$\begin{align*}
m(u) &= \frac{m}{m_2} = \frac{u_0}{\frac{u_0}{c^2}} \\
&= \frac{1}{1 - \frac{v^2}{c^2}}
\end{align*}$$

近似$u_0 << c$ $\Rightarrow$ $\frac{m}{m_2} = u_0 \frac{1}{1 - \frac{v^2}{c^2}} << u$

因此$\frac{m}{m_2} = v$

$m_0 \approx 0$

$m_2 >> v$

$$\begin{align*}
m(ax) &= \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{align*}$$

$m_0$是经典条件，$m(ax)$是粒子的动量，与粒子沿$x$轴的静止测量相比是更大。
Conservation of momentum is valid in relativity provided we write

\[ p = \frac{m_0 \mu}{\sqrt{1 - \mu^2/c^2}} \]

and so \( p_{y_1} + p_{y_2} = 0 \)

Proof

\[ p_{y_1} = \frac{m_0 \mu y_1}{\sqrt{1 - \mu^2 y_1^2/c^2}} \quad p_{y_2} = \frac{m_0 \mu y_2}{\sqrt{1 - (\mu^2 y_2 + \mu^2_2)/c^2}} \]

showing \( p_{y_1} + p_{y_2} = 0 \)

\[ P_{y_2} = -\frac{m_0 \mu_0}{\sqrt{1 - \mu_0^2/c^2}} \]

\[ P_{y_2} = -\frac{m_0 \mu_0}{\sqrt{1 - (\mu_0^2 + \mu_0^2 - \mu_0^2\mu_0^2)/c^2}} \]

\[ \sqrt{1 - (\mu_0^2 - \mu_0^2)/c^2} \]

\[ \sqrt{1 - (1 - \mu_0^2)/c^2} \]

\[ \sqrt{1 - \mu_0^2/c^2} \]

\[ \sqrt{1 - \mu_0^2/c^2} \]

\[ P_{y_2} = -P_{y_1} \]

Buchner 1909

1st experimental confirmation of the dependence of mass on velocity

\( c \) of high velocity \( \rightarrow \) technique similar to the one used by Thomson to measure \( g/m \)

\[ q_m B = m \mu^2 \Rightarrow \mu = \frac{m}{q_m B} \]

prove that \( m(\mu) \)

and that \( c = 2.998 \times 10^8 \text{ m/s} \)
Example 2.1

For what value of \( \mu c \) will the measured mass of an object \( \frac{m_o}{\sqrt{1-\mu^2/c^2}} \) exceed the rest mass by a fraction \( f \)?

\[ f = \frac{1}{\sqrt{1-\mu^2/c^2}} - 1 \Rightarrow \frac{1}{(1-\mu^2/c^2)} = (f+1)^2 \Rightarrow 1-\mu^2/c^2 = \frac{1}{(f+1)^2} \]

\[ \frac{\mu^2}{c^2} = \frac{(f+1)^2 - 1}{(f+1)^2} \Rightarrow \frac{\mu}{c} = \frac{\sqrt{f^2+2f}}{f+1} \Rightarrow \frac{\mu}{c} = \frac{f(f+2)}{f+1} \]

Example 2.2

A high-speed interplanetary probe with a mass \( m = 50,000 \text{ kg} \) has been sent toward Pluto at a speed \( \mu = 0.8c \). What is its momentum as measured by Mission Control on Earth? If, preparatory to landing on Pluto, the probe’s speed is reduced to \( 0.4c \), by how much does its momentum change?

\[ P = \frac{m_0 \mu}{\sqrt{1-\mu^2/c^2}} = \frac{50 \times 10^3 (0.8c)}{\sqrt{1-(0.8)^2}} = 2.0 \times 10^{13} \text{ kg m/s} \]

\[ P_{0.8c} - P_{0.4c} = 2.0 \times 10^{13} - \frac{50 \times 10^3 (0.4c)}{\sqrt{1-(0.4)^2}} = 2.0 \times 10^{13} - 6.5 \times 10^{12} = 1.6 \times 10^{13} \text{ kg m/s} \]