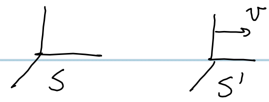
Galilean transformation



$$x' = x - vt \quad t' = t \;\Rightarrow\; \frac{dx'}{dt} = \frac{dx}{dt} - v \;\Rightarrow\; \frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} \;\Rightarrow\; \underbrace{F_x' = F_x}_{m a_x' = m a_x}$$

mass as intrinsic property

Under Lorentz transformation

$\vec{F} = m\vec{a}$ is NOT invariant

because $a_x' \neq a_x$

$$\left.\begin{aligned} x' &= \gamma\,(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right)\end{aligned}\right\} \quad u_x' = \frac{dx'}{dt'} = \frac{(dx/dt - v)}{\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} \;\Rightarrow\; u_x' = \frac{u_x - v}{\left(1 - \frac{v}{c^2}u_x\right)}$$

$$a_x' = \frac{du_x'}{dt'} = \frac{\frac{du_x}{\gamma(dt - v/c^2\,dx)}\left(1 - \frac{v}{c^2}u_x\right) - \left[\left(-\frac{v}{c^2}du_x\right)\Big/\gamma\left(dt - v/c^2\,dx\right)\right](u_x - v)}{\left(1 - \frac{v}{c^2}u_x\right)^2}$$

$$= \frac{\left[\frac{a_x}{\gamma\cancel{(1 - v/c^2\,u_x)}}\cancel{(1 - v/c^2\,u_x)}\right] + \frac{v}{c^2}\frac{a_x}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}(u_x - v)}{\left(1 - \frac{v}{c^2}u_x\right)^2}$$

$$= \frac{\frac{a_x}{\gamma\left(1 - v/c^2\,u_x\right)}\left[\underbrace{\left(1 - \cancel{v/c^2\,u_x}\right) + \cancel{\frac{v}{c^2}u_x} - \frac{v^2}{c^2}}_{\frac{1}{\gamma^2}}\right]}{\left(1 - \frac{v}{c^2}u_x\right)^2} = \boxed{\frac{a_x}{\gamma^3\left(1 - \frac{v u_x}{c^2}\right)^3}}$$

F=ma would increase the velocity indefinitely. Relativistic mass(inertia) increases with the speed, preventing that v ever becomes > than c.

Relativistic Mass

Lorentz transf. ⇒ need modification of the equations of mechanics so that they remain **INVARIANT** under the transformation from one inertial frame to another

⇒ **RELATIVISTIC mechanics**

Newton's 2nd law in the form $\vec{F} = m\vec{a}$ is not relativistically invariant

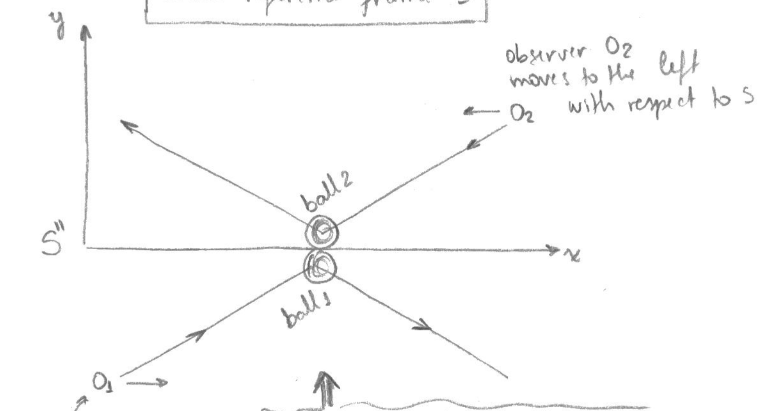
but

$\vec{F} = \dfrac{d\vec{p}}{dt}$ is relativistically invariant if the relativistic momentum $\vec{p}$ is used

$$\boxed{\vec{p} = \underbrace{m(v)}_{\text{mass is a function of } v}\,\vec{v}}$$

Thought experiment

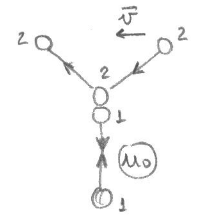
From reference frame S''



conservation of momentum

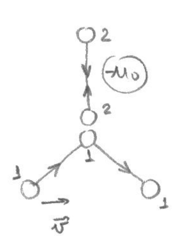
elastic collision

From the reference frame of O1 } (S)

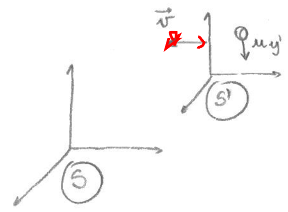


$\vec{v}$: velocity of O2 with respect to O1

From the reference frame of O2 (S')



$\vec{v}$: velocity of O1 with respect to O2



→) From the reference frame of O_1 (S)

Ball 1 moves along the y axis with velocity $u_{y_1} = u_0$

Ball 2 has an x and y components of its velocity

$$\text{Ball 2}\begin{cases} u_{x_2} = \dfrac{u_{x_2}' + v}{1 + \frac{v}{c^2} u_{x_2}'} \xrightarrow{u_x' = 0} u_{x_2} = v \\[2ex] u_{y_2} = \dfrac{u_{y_2}'}{\gamma\left(1 + \frac{v}{c^2} u_{x_2}'\right)} \xrightarrow{u_y' = -u_0} u_{y_2} = \ominus \dfrac{u_0}{\gamma} \end{cases}$$

u_{y_2} is smaller than u_{y_1}, because time taken for ② to travel in S (O_1) is greater than measured in S' (O_2)

$$\underbrace{\begin{array}{ll} \text{Before collision} & u_{y_1} = u_0 \\ & u_{y_2} = -\dfrac{u_0}{\gamma} \end{array} \qquad \begin{array}{ll} \text{After collision} & u_{y_1} = -u_0 \\ & u_{y_2} = \dfrac{u_0}{\gamma} \end{array}}$$

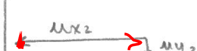
to guarantee conservation of momentum in S ⇒ $m(v)$

$$\underset{\text{(before collision)}}{m(u_0)\, u_0 - m(u)\, u_{y_2}} = \underset{\text{(after collision)}}{-m(u_0)\, u_0 + m(u)\, u_{y_2}}$$

$$\frac{m(u)}{m(u_0)} = \frac{u_0}{u_{y_2}} = \frac{u_0}{u_0\sqrt{1 - v^2/c^2}}$$

Approximation : $u_0 << v \implies u_{y_2} = u_0\sqrt{1 - \frac{v^2}{c^2}} << v$

since $u_{x_2} = v$



$\boxed{u_2 \simeq v}$

$u_0 \simeq 0$

$u_2 \simeq v$

$$\boxed{m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}}$$

m_0 is the classically measured mass of the particle (mass of the particle <u>at rest</u>)

mass of an object in motion with respect to an observer is <u>LARGER</u> than measured when it is at rest

→) Conservation of momentum is valid in relativity provided we write

$$\boxed{\vec{p} = \frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}}$$

mass

$$m(u) = \frac{m_0}{\sqrt{1 - u^2/c^2}}$$

→) in the reference frame of O_1 (S) $\left\{ \begin{array}{l} p_{y1} + p_{y2} \text{ (before collision)} \\ \text{and} \\ p_{y1} + p_{y2} \text{ (after collision) only changes signs, so for} \end{array} \right.$ momentum to be conserved $p_{y1} + p_{y2} = 0$

Proof

$$p_{y1} = \frac{m_0 u_0}{\sqrt{1 - u_0^2/c^2}} \qquad p_{y2} = \frac{m_0 u_{y2}}{\sqrt{1 - (u_{x2}^2 + u_{y2}^2)/c^2}}$$

and so $p_{y1} + p_{y2} = 0$

showing it

$$p_{y2} = -\frac{m_0 u_0 \sqrt{1 - v^2/c^2}}{\sqrt{1 - \left(v^2 + u_0^2 - u_0^2 \frac{v^2}{c^2}\right)/c^2}} = \frac{-m_0 u_0 \sqrt{1 - v^2/c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right) - \frac{u_0^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)}} = \frac{-m_0 u_0 \cancel{\sqrt{1 - v^2/c^2}}}{\cancel{\sqrt{1 - v^2/c^2}} \sqrt{1 - u_0^2/c^2}}$$

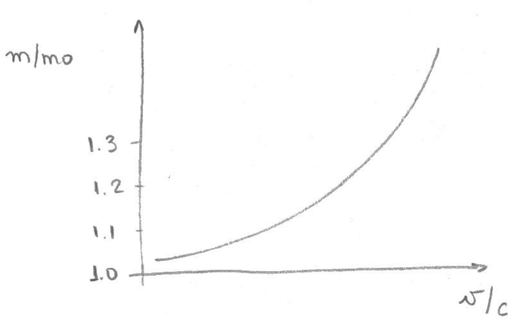
$$\boxed{p_{y2} = -p_{y1}}$$

Bucherer 1909

1st experimental confirmation of the dependence of mass on velocity

e^- of high velocity → technique similar to the one used by Thomson to measure q/m

↳ $quB = m\frac{u^2}{R} \Rightarrow \frac{q}{m} = \frac{u}{RB}$



prove that $m(v)$

and that $c = 2.998 \times 10^8$ m/s

Example 2-1

For what value of (u/c) will the measured mass of an object $\frac{m_0}{\sqrt{1-u^2/c^2}}$ exceed the rest mass by a fraction f?

↳ that is, $\frac{m_0}{\sqrt{1-u^2/c^2}} = m_0 + m_0 f$

$$f = \frac{1}{\sqrt{1-u^2/c^2}} - 1 \Rightarrow \frac{1}{(1-u^2/c^2)} = (f+1)^2 \Rightarrow 1 - u^2/c^2 = \frac{1}{(f+1)^2}$$

$$u^2/c^2 = \frac{(f+1)^2 - 1}{(f+1)^2} \Rightarrow u/c = \frac{\sqrt{f^2+2f}}{(f+1)} \Rightarrow \boxed{u/c = \frac{\sqrt{f(f+2)}}{f+1}}$$

Example 2-2

A high-speed interplanetary probe with a mass m = 50,000 kg has been sent toward Pluto at a speed u = 0.8c. What is its momentum as measured by Mission Control on Earth? If, preparatory to landing on Pluto, the probe's speed is reduced to 0.4c, by how much does its momentum change?

$$p_{0.8c} = \frac{m_0 u}{\sqrt{1-u^2/c^2}} = \frac{50\times10^3\,(0.8c)}{\sqrt{1-(0.8)^2}} = \boxed{2.0\times10^{13}\ \text{kg m/s}}$$

$$p_{0.8c} - p_{0.4c} = 2.0\times10^{13} - \frac{50\times10^3\,(0.4c)}{\sqrt{1-(0.4)^2}} = 2.0\times10^{13} - 6.5\times10^{12} = \boxed{1.6\times10^{13}\ \text{kg m/s}}$$