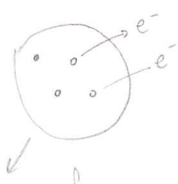
Nuclear Model

1910 - Experiments (scattering of x rays by atoms, photoelectric effect, etc) had been shown that atoms contain e^-. Since atoms are neutral, they must also contain positive charges. Because m_{e^-} is so small, $\Rightarrow$ most of the atom mass must be related to the positive charges

Thomson's Model: e^- were located within a continuous distribution of + charge

+ charge distribution – spherical, $r \sim 10^{-10}$ m (radius of the atom)

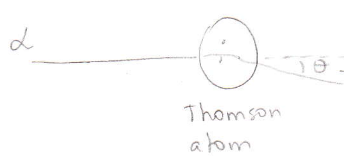


"plum pudding" model : e^- distributed through the sphere

sphere of + charges

Problem 1 :
- At lowest energy: e^- would be fixed at equilibrium position according to electrostatic forces $\rightarrow$ no possible configuration found
- In excited atom : e^- would vibrate
  - ↳ continuous emission of radiation
  - BUT not observed, and loss of energy – e^- would spiral into the nucleus

Problem 2 : could not explain large scattering angles of α particles



$\rightarrow$ small angle

$\rightarrow$) ⊕ charge distributed cannot provide intense Coulomb repulsion to explain large deflection (even $\theta = 180^\circ$ is seen)

$\rightarrow$) e^- are too small $\Rightarrow$ only small deflection

Rutherford's Model

↳ ⊕ charge is not spread throughout atom but is concentrated in a small region at the center

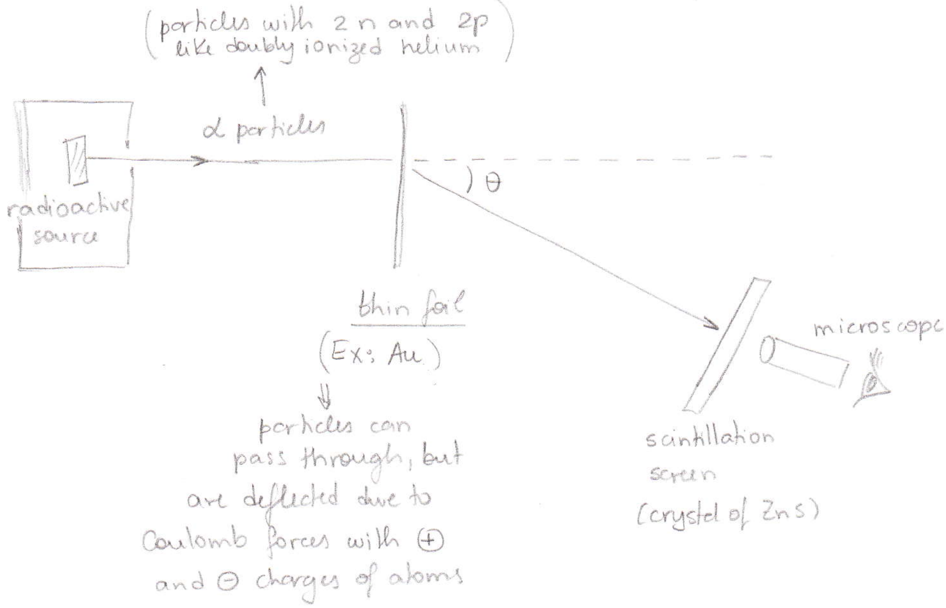
NUCLEUS $\rightarrow$ from 10^{-15} m to 10^{-14} m (1 fm – 10 fm)

1 fermi / femtometer } 1 fm = 10^{-15} m

solve Problem 2, but not Problem 1

↳ need Bohr's model

The experiment



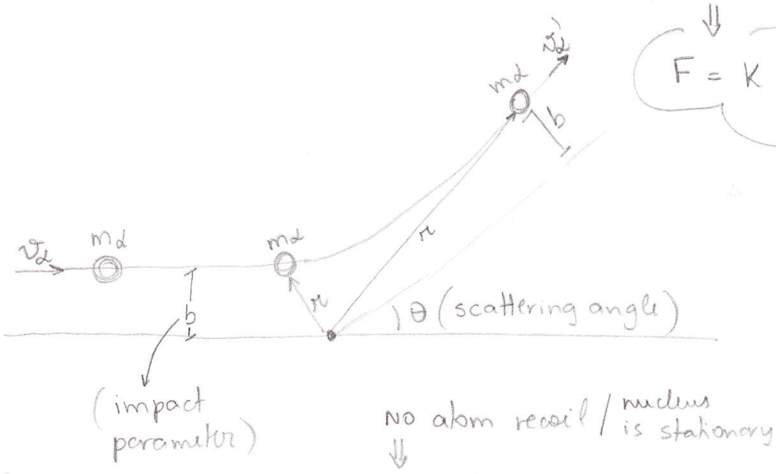
Rutherford's assumptions to obtain the theorical angular distribution for the scattered α particles (confirmed by his experiment):

For large angles:

- scattering due to repulsive Coulomb between α and ⊕ charged nucleus
- m of nucleus is so large compared to α ⇒ no recoil (heavy atom)
  - ↳ nucleus is a point charge / fixed
- α do not penetrate nuclear region
  - ↳ nucleus and α act as point charges

⇓

$$F = K \frac{q_\alpha Q}{r^2}$$



No atom recoil / nucleus is stationary

⇓

$v_\alpha = v_\alpha'$

$\frac{1}{2} m v_\alpha^2 = \frac{1}{2} m v_\alpha'^2$

$$\boxed{b = \frac{K q_\alpha Q}{m_\alpha v^2} \cot(\theta/2)}$$

⇓

smaller b ⇒ larger θ

The hyperbolic Rutherford trajectory

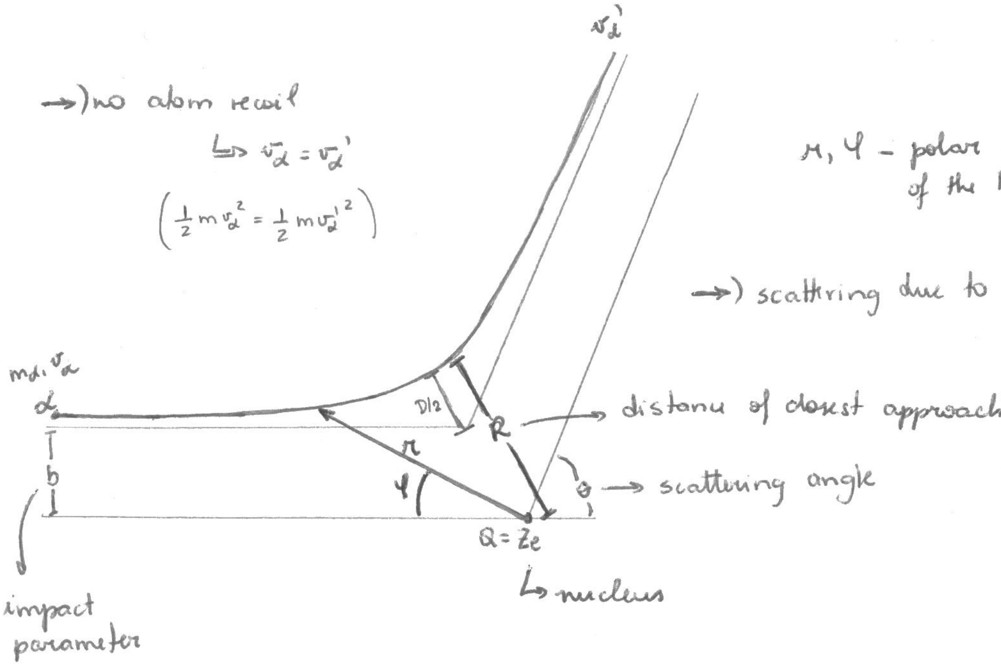
→) no atom recoil

↳ $v_\alpha = v_\alpha'$

$\left(\frac{1}{2} m v_\alpha^2 = \frac{1}{2} m v_\alpha'^2\right)$

r, φ – polar coordinates of the hyperbola

→) scattering due to repulsive Coulomb $F = \frac{k\, q_\alpha Q}{r^2}$



$$\boxed{\frac{1}{r} = \frac{1}{b}\sin\varphi + \frac{D}{2b^2}(\cos\varphi - 1)}$$ ← trajectory of the α particle

$$\boxed{D \equiv \frac{k\, q_\alpha Q}{\frac{1}{2} m_\alpha v^2}}$$ ← distance of closest approach in a head-on collision ($b=0$)

$\frac{1}{2} m_\alpha v_\alpha^2 = \frac{k\, q_\alpha Q}{D}$ ⇐ distance where particle stops and reverses its direction of motion; initial kinetic energy = potential energy

→) For $b > 0$: particle does not stop, distance of closest approach R is larger than D

→) The scattering angle θ corresponds to $\pi - \varphi$ as $r \to \infty$

$$\theta = \pi - \varphi$$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$
$\sin(a+b) = \sin a \cos b + \sin b \cos a$

$$r \to \infty \Rightarrow \frac{1}{r} \to 0 \Rightarrow -\frac{1}{b}\sin(\pi - \theta) = \frac{D}{2b^2}\left(\cos(\pi-\theta) - 1\right)$$

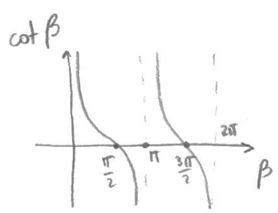
$$-\frac{1}{\cancel{b}}\sin\theta = \frac{D}{2b^{\cancel{2}}}\left(-\cos\theta - 1\right)$$

$$b = (D/2)\,\frac{(2\cos^2\theta/2 - 1 + 1)}{2\sin\theta/2\,\cos\theta/2}$$

HW → show all steps

$$\boxed{b = \frac{D}{2}\cot(\theta/2)}$$

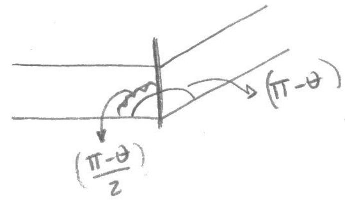
$0 \le \theta \le \pi$
$\theta = 0 \Rightarrow b \to \infty$
$\theta = \pi \Rightarrow b \to 0$



HW show all the steps

Exercise : Evaluate R, the distance of closest approach of the particle to the center of the nucleus

$r = R$ when $\varphi = (\pi - \theta)/2$



$$\frac{1}{R} = \frac{1}{b} \sin\left(\frac{\pi-\theta}{2}\right) + \frac{D}{2b^2}\left[\cos\left(\frac{\pi-\theta}{2}\right) - 1\right]$$

$$b = (D/2)\cot(\theta/2) = (D/2)\tan\left(\frac{\pi-\theta}{2}\right)$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin(\pi/2 - x)}{\cos(\pi/2 - x)} = \frac{\cos x}{\sin x} = \cot x$$

⋮

$$\boxed{R = \frac{D}{2}\left[1 + \frac{1}{\sin(\theta/2)}\right]}$$

- As $\theta \to \pi$ (b = 0, head-on collision)

  ↳ $\boxed{R \to D}$

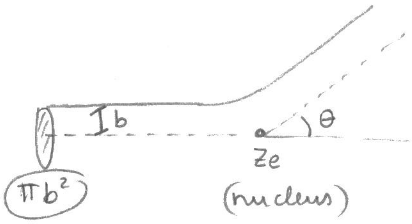
- As $\theta \to 0$ ($b \to \infty$, no deflection)

  ↳ $\boxed{R \to \infty}$

But b is not known in advance. Experiment measures number of detected α particles and θ.

→) I_0 : particles per second per unit area (intensity of incident α particle beam)

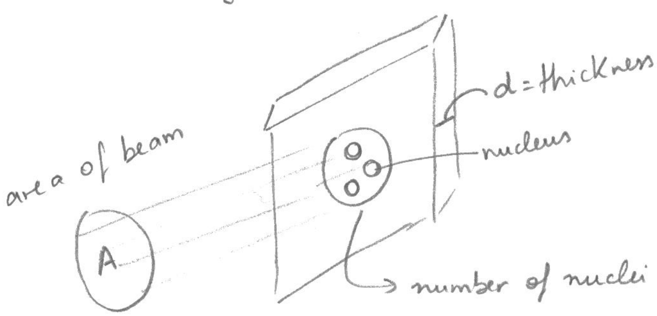
→) $(\pi b^2) I_0$: number of particles scattered per second by one nucleus through angles $\geq \theta$



cross section

for scattering through angles greater than θ

→) $(\pi b^2) I_0\, n\, d A$: total number of particles scattered (with angles greater than θ) per second for a beam of area A through a foil of thickness d containing n nuclei per unit volume



number of nuclei seen by beam : $\boxed{nAd}$

$$n = \frac{N}{V}$$

$$\left.\begin{array}{l} N \longrightarrow M \\ N_A \longrightarrow M(1\,mol) \end{array}\right\} \quad N = \frac{M\, N_A}{M(1\,mol)}$$

$$n = \frac{M}{V}\,\frac{N_A}{M(1\,mol)} = \boxed{\rho\,\frac{N_A}{M(1\,mol)}}$$

$$f = \frac{(\pi b^2) I_0\, n\, d A}{I_0 A}$$ = fraction of the number of α particles scattered through angles greater than θ

$I_0 A$ → number of α particles incident per second

Experimentally we can measure f and verify if it agrees with the theory

$$\boxed{f = \pi b^2 n d}$$

$$\boxed{n = \frac{\rho N_A}{M(/mol)}}$$

$$\boxed{b = \frac{k q_\alpha Q}{m_\alpha v^2} \cot(\theta/2)}$$

For example:

for an incident beam of α particles with Kinetic energy 5 MeV, the fraction f of scattred particles from a gold foil with $\theta \geq 90°$ was verified to be $\rho_{Au} = 19.3\, g/cm^3$

(Z = 79)

10^{-6} m thick

$\boxed{f \sim 10^{-4}}$ ↳ which agrees with the thoretical calculation as shown below

$$\circ)\ n = \frac{\rho N_A}{M(/mol)} = \frac{(19.3\, g/cm^3)(6.02 \times 10^{23}\, atoms/mol)}{(197\, g/mol)} = 5.9 \times 10^{28}\, atoms/m^3$$

$$\circ)\ b = \frac{K q_\alpha Q}{m_\alpha v^2} \cot(\theta/2) = \frac{k(2e)(79e)}{2(5 \times 10^6\, eV)} \overbrace{\cot(45°)}^{1} = 2.28 \times 10^{-14}\, m$$

$$k e^2 = 1.440\, eV \cdot nm$$

$$\Rightarrow f = \pi (2.28 \times 10^{-14} m)^2 (5.9 \times 10^{28}\, atoms/m^3)(10^{-6} m)$$

$$f = 9.6 \times 10^{-5} \simeq 10^{-4}$$

Rutherford also derived an expression for the number of α particles ΔN scattered at any angle θ:

$$\boxed{\Delta N = \left(\frac{I_0 A_{sc} n d}{r_{fd}^2}\right)\left(\frac{k Z e^2}{2 E_k}\right)\frac{1}{\sin^4(\theta/2)}}$$

↳ Kinetic energy of incident particles

geometry of the detector

foil - detector distance

Exercise: α particles from ^{226}Ra are scattered at $\theta = 45°$ from a silver foil and 450 particles are counted each minute at the detector.

If the detector is moved to observe particles at 90°, how many will be counted per minute?

$$45° \Rightarrow \frac{\Delta N}{min} = 450 = \text{Const} \sin^{-4}(45°/2) \Rightarrow \text{Const} = \underline{450} \sin^4(45°/2)$$

$$90° \Rightarrow \frac{\Delta N}{min} = \left(450 \sin^4(45°/2)\right) \sin^{-4}(90°/2) \simeq 39 \text{ particles/min}$$

$\hookrightarrow$ Obs: (for very high energy, particles penetrate nucleus $\Rightarrow$ $F = \frac{k q_\alpha Q}{r^2}$ is no longer valid and data don't agree with expression for ΔN)

Size of the nucleus

For 180° the collision is nearly 'head-on'
The closest approach of α to the nucleus is an experimental upper limit on the size of the nucleus

Conservation of energy: $\underbrace{(V+E)_{\text{large } r}}_{\frac{1}{2} m_\alpha v^2} = \underbrace{(V+E)_{r_{\text{nucleus}}}}_{\frac{k q_\alpha Q}{r_d}}$

$$\boxed{r_d = \frac{k q_\alpha Q}{\frac{1}{2} m_\alpha v^2}} \sim 3 \times 10^{-14} \text{m for 7.7 MeV } \alpha \text{ particles}$$

Exercise:

$r_{Au} = 6.6$ fm. What K_α is necessary for this measurement?

$$K_\alpha = \frac{k q_\alpha Q}{r_{Au}} = \frac{(9 \times 10^9 \text{ Nm}^2/c^2)(2)(79)(1.60 \times 10^{-19} C)^2}{6.6 \times 10^{-15} m}$$

$$K_\alpha = 5.52 \times 10^{-12} J = \boxed{34.5 \text{ MeV}}$$