Nuclear Model

1910- Experiments (scattering of x-rays by atoms, photoelectric effect, etc.) had
been shown that atoms contain e-. Since atoms are neutral, they must
also contain positive charges. Because e- is so small, most of the
atom mass must be related to the positive charges.

Thomson's Model: e- were located within a continuous distribution of + charge
+ charge distribution - spherical, \( r \approx 10^{-10} \text{m} \) (radius of the atom)

"plum pudding" model: e- distributed throughout the sphere.

Problem 1:

- At lowest energy: e- would be fixed at equilibrium position
- According to electrostatic forces - no possible configuration found.
- In excited atom: e- would vibrate
- Continuous emission of radiation
- But not observed, and loss of energy - e- would spiral into the nucleus.

Problem 2:

Could not explain large scattering angles of alpha particles

\( \alpha \) - Charge distributed cannot provide intense Coulomb repulsion to explain large deflection

At \( \theta = 180^\circ \) is seen

\( \Rightarrow \) e- are too small, so only small deflection.

Rutherford's Model

\( \Rightarrow \) + charge is not spread throughout atom

but is concentrated in a small region at the center

\[ \text{NUCLEUS} \rightarrow \text{from } 10^{-15} \text{m to } 10^{-14} \text{m} \ (1 \text{ fm} \rightarrow 10 \text{ fm}) \]

1 fermi \( \rightarrow \) 1 fermi = \( 10^{-15} \text{m} \)

Solve Problem 2,

but not Problem 1

\( \rightarrow \) need Bohr's model
The experiment

\[(\text{particles with } 2\text{ }n\text{ and } 2p)\]

\[\text{with doubly ionized helium}\]

\[\alpha\text{ particles}\]

\[\text{radiactive source}\]

\[\text{thin foil}\]

\[(\text{Ex: Au})\]

\[\text{particles can pass through, but are deflected due to Coulomb forces with } +\text{ and } \theta\text{ charges of atoms}\]

\[\text{scintillation screen}\]

\[\text{(crystal of Zns)}\]

Rutherford's assumptions to obtain the theoretical angular distribution for the scattered \(\alpha\) particles (confirmed by his experiment):

For large angles:

1. Scattering due to repulsive Coulomb between \(\alpha\) and \(+\) charged nucleus
2. Mass of nucleus is so large compared to \(\alpha\) = no recoil (heavy atom)

\[\Rightarrow \text{nucleus is a point charge / fixed}\]

3. \(\alpha\) do not penetrate nuclear region

\[\Rightarrow \text{nucleus and } \alpha\text{ act as point charges}\]

\[F = K \frac{q_\alpha q_\theta}{r^2}\]

\[b = K \frac{q_\alpha q_\theta}{m_\alpha v^2} \cos(\theta/2)\]

\[\text{smaller } b \Rightarrow \text{larger } \theta\]
The hyperbolic Rutherford trajectory

\[ -m_0 \gamma' \alpha = m_0 \gamma' \alpha' \]
\[ \left( \frac{1}{2} m_0 \gamma'^2 = \frac{1}{2} m_0 \gamma^2 \right) \]

\[ R, \theta - \text{polar coordinates of the hyperbola} \]

\[ F = \frac{k q_1 q_2}{r^2} \]

\[ \text{scattering due to repulsive Coulomb force} \]

\[ R = b \theta \]

\[ \text{impact parameter} \]

\[ \frac{1}{n} = \frac{1}{b} \sin \theta + \frac{D}{2b^2} \cos (\pi - \theta - 1) \]

\[ \text{- trajectory of the } \alpha \text{ particle} \]

\[ D = \frac{k q_1 q_2}{\frac{1}{2} m_0 \gamma^2} \]

\[ \text{- distance of closest approach in a head-on collision (} b = 0 \text{)} \]

\[ \frac{1}{2} m_0 v_0^2 = \frac{k q_1 q_2}{D} \]

\[ \left\{ \begin{array}{l}
\text{distance where particle slopes and}
\text{reverses its direction of motion}
\text{inital Kinetic energy = potential energy}
\end{array} \right. \]

\[ \rightarrow \text{For } b > 0 \text{ : particle does not stop, distance of closest approach } R \text{ is larger than } D \]

\[ \rightarrow \text{The scattering angle } \theta \text{ corresponds to } \pi - \theta \text{ as } n \to \infty \]

\[ \theta = \pi - \theta \]

\[ n \to \infty \Rightarrow \frac{1}{n} \to 0 \Rightarrow -\frac{1}{b} \sin (\pi - \theta) = \frac{D}{2b^2} \left( \cos (\pi - \theta) - 1 \right) \]

\[ -\frac{1}{b} \sin \theta = \frac{D}{2b^2} \left( -\cos \theta - 1 \right) \]

\[ b = \left( \frac{D}{2} \right) \left( \frac{2 \sin^2 \theta / 2 - 1 + 1}{2 \sin \theta/2 \cos \theta/2} \right) \]

\[ b = \frac{D}{2} \cot \left( \frac{\theta}{2} \right) \]

\[ \cot \beta \]

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HW → show all steps

\[ 0 \leq \theta \leq \pi \]

\[ \theta = 0 \Rightarrow b \to \infty \]

\[ \theta = \pi \Rightarrow b \to 0 \]
Exercise: Evaluate $R$, the distance of closest approach of the particle to the center of the nucleus.

$r = R$ when $\phi = (\pi - \theta)/2$

\[
\frac{1}{R} = \frac{1}{b} \sin \left( \frac{\pi - \theta}{2} \right) + \frac{D}{2b^2} \left[ \cos \left( \frac{\pi - \theta}{2} \right) - 1 \right]
\]

\[b = \left( \frac{D}{2} \right) \cot \left( \frac{\theta}{2} \right) = \left( \frac{D}{2} \right) \tan \left( \frac{\pi - \theta}{2} \right)\]

\[R = \frac{D}{2} \left[ 1 + \frac{1}{\sin(\theta/2)} \right]\]

1. As $\theta \to \pi$ ($b = 0$, head-on collision)

\[\Rightarrow R \to D\]

2. As $\theta \to 0$ ($b \to \infty$, no deflection)

\[\Rightarrow R \to \infty\]
But $b$ is not known in advance. Experiment measures number of detected $d$ particles and $\Theta$

$\rightarrow$ $I_0$: particles per second per unit area (intensity of incident $d$ particle beam)

$\rightarrow$ $(\pi b^2) I_0$: number of particles scattered per second by one nucleus through angles $\Theta$

Cross section for scattering through angles greater than $\Theta$

$\rightarrow$ $(\pi b^2) I_0 n \, dA$: total number of particles scattered per second for a beam of area $A$ through a foil of thickness $d$ containing $n$ nuclei per unit volume

$n = \frac{N}{V}$

$N \rightarrow M$

$N_A \rightarrow M \, (1 \text{mol})$

$\{ N = \frac{M \, N_A}{M(1 \text{mol})} \}$

$n = \frac{M}{V} \, \frac{N_A}{M(1 \text{mol})} = \frac{\rho \, N_A}{M(1 \text{mol})}$

$f = \frac{(\pi b^2) I_0 \, n \, dA}{I_0 \, A}$: fraction of the number of $d$ particles scattered through angles greater than $\Theta$ incident per second
Experimentally we can measure \( f \) and verify if it agrees with the theory

\[
f = \pi b^2 n d
\]

\[
n = \frac{\rho NA}{M \text{ (mol)}}
\]

\[
b = \frac{k q_0 Q \cot(\theta/2)}{m_0 v^2}
\]

For example:

for an incident beam of \( \alpha \) particles with kinetic energy \( 5 \text{ MeV} \), the fraction \( f \) of scattered particles from a gold foil with \( \theta \geq 90^\circ \) was verified to be

\[
f \approx 10^{-4}
\]

\( 10^{-6} \text{ m thick} \)

\( Z = 79 \)

\( \Rightarrow \) which agrees with the theoretical calculation as shown below

\( \rho = 19.3 \text{ g/cm}^3 \)

\( 6.02 \times 10^{23} \text{ atoms/mol} \)

\( 197 \text{ g/mol} \)

1. \( n = \frac{\rho NA}{M \text{ (mol)}} = \frac{(19.3 \text{ g/cm}^3)(6.02 \times 10^{23} \text{ atoms/mol})}{197 \text{ g/mol}} = 5.9 \times 10^{28} \text{ atoms/m}^3 \)

2. \( b = \frac{k q_0 Q \cot(\theta/2)}{m_0 v^2} = \frac{k (2e)(79e)}{2 (5 \times 10^6 \text{ eV})} \cot(45^\circ) = 2.28 \times 10^{-14} \text{ m} \)

\( k e^2 = 1.44 \text{ eV nm} \)

\( \Rightarrow f = \pi (2.28 \times 10^{-14} \text{ m})^2 (5.9 \times 10^{28} \text{ atoms/m}^3) (10^{-6} \text{ m}) \)

\( f = 9.6 \times 10^{-5} \approx 10^{-4} \)

Rutherford also derived an expression for the number of \( \alpha \) particles \( \Delta N \) scattered at any angle \( \theta \):

\[
\Delta N = \left( \frac{I_0 A_s n d}{\kappa_{fd}} \right) \left( \frac{k Z e^2}{2 (E_k)} \right) \frac{1}{\sin^4(\theta/2)}
\]

\( \Rightarrow \) Kinetic energy of incident particles

geometry of

the detector

foil - detector distance
Exercise: α particles from $^{226}\text{Ra}$ are scattered at $\theta = 45^\circ$ from a silver foil and 450 particles are counted each minute at the detector.

If the detector is moved to observe particles at $90^\circ$, how many will be counted per minute?

\[
45^\circ \Rightarrow \frac{\Delta N}{\text{min}} = 450 = \text{Const} \sin^{-4} \left(\frac{45^\circ}{2}\right) \Rightarrow \text{Const} = \frac{450}{\sin^4 \left(\frac{45^\circ}{2}\right)}
\]

\[
90^\circ \Rightarrow \frac{\Delta N}{\text{min}} = \left(450 \sin^4 \left(\frac{45^\circ}{2}\right)\right) \sin^{-4} \left(\frac{90^\circ}{2}\right) \approx 39 \text{ particles/min}
\]

L-Obs: (For very high energy, particles penetrate nucleus =)

E $\geq E_{\text{cut}}$ is no longer valid and data don't agree with expression for $\Delta N$.

**Size of the nucleus**

For $180^\circ$ the collision is nearly 'head-on'

the closest approach of $\alpha$ to the nucleus is an experimental upper limit on the size of the nucleus.

**Conservation of energy:**

\[
\left(\frac{\sqrt{V + E}}{m_n}\right)_{\text{large}} = \left(\frac{\sqrt{V + E}}{m_{\text{nucleus}}}\right)
\]

\[
\frac{1}{2} m_n v^2 = \frac{k \alpha Q}{4 \pi d}
\]

\[
R_d = \frac{k \alpha Q}{\frac{1}{2} m_n v^2}
\]

\[
R_d \approx 3 \times 10^{-14} \text{ m for 7.7 MeV } \alpha \text{ particles}
\]

Exercise:

$K_{\text{max}} = 6.6 \text{ fm}$ - what $K_\alpha$ is necessary for this measurement?

\[
K_\alpha = \frac{k \alpha Q}{M_{\text{max}}} = \left(9 \times 10^9 \text{ N m}^2/\text{C}^2\right) \left(2\right) \left(\text{Z}^2\right) \left(1.60 \times 10^{-19} \text{ C}\right)^2 \\
6.6 \times 10^{-15} \text{ m}
\]

\[
K_\alpha = 5.52 \times 10^{-12} \text{ J} = 34.5 \text{ MeV}
\]