# Assignment 09

The concentration of salt x in a homemade soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5\,x$$

At the initial time, t =0 , the salt concentration in the tank is 50 g/L.

a) Using *Mathematica,* what is the salt concentration after 3 minutes?

b) Using Euler' method and a step size of h = 0.75 min , what is the salt concentration after 3 minutes?

c) Same as above, but using Taylor method with the expansion to 4th order.

d) Same as above, but using Runge-Kutta 4th order method.

e) What is the absolute relative errors between the solution from each one of the methods used above and the "exact" solution from *Mathematica*.

f) Using Runge-Kutta 4th order method, find a value of the step h, which gives an absolute relative error between 1% and 2%.

## a) *Mathematica*

```
Clear[x, t, sol, solut, xExact];
sol = DSolve[{x'[t] == 37.5 - 3.5 x[t], x[0] == 50.}, x[t], t];
solut = sol〚1, 1, 2〛;
xExact = solut /. t → 3.
10.7154
```

## b) Euler

```
Clear[xo, to, h, howmany]
to = 0.;
xo = 50.;
h = 0.75;
howmany = (3 - to) / h;

Clear[dx]
dx = 37.5 - 3.5 x;

Clear[tanX];
Do[
  tanX = dx /. {t → to, x → xo};
  to = to + h;
  xo = xo + tanX h;

  , {k, 1, howmany}];

Clear[xEuler];
xEuler = xo
```

```
284.65
```

## c) Taylor

```
Clear[xo, to, h, howmany]
to = 0.;
xo = 50.;
h = 0.75;
howmany = (3 - to) / h;

Clear[x1, x2, x3, x4];
x1 = 37.5 - 3.5 x[t];
x2 = D[x1, t] /. x'[t] → x1;
x3 = D[x2, t] /. x'[t] → x1;
x4 = D[x3, t] /. x'[t] → x1;

Clear[TaylorX, Rx1, Rx2, Rx3, Rx4];
Do[

  Rx1 = x1 /. {t → to, x[t] → xo};
  Rx2 = x2 /. {t → to, x[t] → xo};
  Rx3 = x3 /. {t → to, x[t] → xo};
  Rx4 = x4 /. {t → to, x[t] → xo};

  TaylorX = xo + h Rx1 + (1 / 2) h^2 Rx2 + (1 / 6) h^3 Rx3 + (1 / 24) h^4 Rx4;

  to = to + h;
  xo = TaylorX;

  , {k, 1, howmany}];

Clear[xTaylor];
xTaylor = xo

25.5586
```

## d) Runge - Kutta

```
Clear[xo, to, h, howmany]
to = 0.;
xo = 50.;
h = 0.75;
howmany = (3 - to) / h;

Clear[f, x];
f = 37.5 - 3.5 x;

Clear[ k1, k2, k3, k4];
Do[
 k1 = f /. {t → to , x → xo};
 k2 = f /. {t → to + h / 2 , x → xo + k1 h / 2};
 k3 = f /. {t → to + h / 2 , x → xo + k2 h / 2};
 k4 = f /. {t → to + h , x → xo + k3 h};

 to = to + h;
 xo = xo + (k1 + 2 k2 + 2 k3 + k4) h / 6;
 , {k, 1, howmany}]

Clear[xRK];
xRK = xo
```

```
25.5586
```

## e) Errors

```
Print["Error between exact and Euler ", 100 Abs[(xExact - xEuler) / xExact] , "%"]

Print["Error between exact and Euler ", 100 Abs[(xExact - xTaylor) / xExact] , "%"]

Print["Error between exact and Euler ", 100 Abs[(xExact - xRK) / xExact] , "%"]
```

```
Error between exact and Euler 2556.46%
Error between exact and Euler 138.522%
Error between exact and Euler 138.522%
```

### f) Runge-Kutta with error between 1% and 2%

```
Clear[xo, to, h, howmany]
to = 0.;
xo = 50.;
h = 0.55;
howmany = (3 - to) / h;

Clear[f];
f = 37.5 - 3.5 x;

Clear[ k1, k2, k3, k4];
Do[
 k1 = f /. {t → to , x → xo};
 k2 = f /. {t → to + h / 2 , x → xo + k1 h / 2};
 k3 = f /. {t → to + h / 2 , x → xo + k2 h / 2};
 k4 = f /. {t → to + h , x → xo + k3 h};

 to = to + h;
 xo = xo + (k1 + 2 k2 + 2 k3 + k4) h / 6;
 , {k, 1, howmany}]

Clear[xRK];
xRK = xo;
Print["Error between exact and Euler ", 100 Abs[(xExact - xRK) / xExact] , "%"]
```

```
Error between exact and Euler 1.05789%
```

A polluted lake has an initial concentration of a bacteria of $10^7$ parts/$m^3$, while the acceptable level is only $5\times10^6$ parts/$m^3$ . The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06\,C = 0$$

a) Using *Mathematica,* what is the concentration of the pollutant after 7 weeks?

b) Using Euler' method and a step size of h = 3.5 weeks , find the concentration of the pollutant after 7 weeks.

c) Same as above, but using Taylor method with the expansion to 4th order.

d) Same as above, but using Runge-Kutta 4th order method.

e) What is the absolute relative errors between the solution from each one of the methods used above and the “exact” solution from *Mathematica*.

### *a) Mathematica*

```
Clear[x, t, sol, solut, cExact];
sol = DSolve[{CC'[t] == -0.06 CC[t], CC[0] == 10^7}, CC[t], t];
solut = sol[[1, 1, 2]];
cExact = solut /. t → 7.
```

$6.57047 \times 10^{6}$

### b) Euler

```
Clear[CCo, to, h, howmany]
to = 0.;
CCo = 10^7;
h = 3.5;
howmany = (7 - to) / h;

Clear[dx]
dCC = -0.06 CC;

Clear[tanCC];
Do[
  tanCC = dCC /. {t → to, CC → CCo};
  to = to + h;
  CCo = CCo + tanCC h;

  , {k, 1, howmany}];

Clear[cEuler];
cEuler = CCo
```

$6.241 \times 10^{6}$

## c) Taylor

```
Clear[CCo, to, h, howmany]
to = 0.;
CCo = 10^7;
h = 3.5;
howmany = (7 - to) / h;

Clear[c1, c2, c3, c4];
c1 = -0.06 CC[t];
c2 = D[c1, t] /. CC'[t] → c1;
c3 = D[c2, t] /. CC'[t] → c1;
c4 = D[c3, t] /. CC'[t] → c1;

Clear[TaylorX, Rc1, Rc2, Rc3, Rc4];
Do[

  Rc1 = c1 /. {t → to, CC[t] → CCo};
  Rc2 = c2 /. {t → to, CC[t] → CCo};
  Rc3 = c3 /. {t → to, CC[t] → CCo};
  Rc4 = c4 /. {t → to, CC[t] → CCo};

  TaylorC = CCo + h Rc1 + (1 / 2) h^2 Rc2 + (1 / 6) h^3 Rc3 + (1 / 24) h^4 Rc4;

  to = to + h;
  CCo = TaylorC;

  , {k, 1, howmany}];

Clear[cTaylor];
cTaylor = CCo
```

$6.57052 \times 10^{6}$

## d) Runge - Kutta

```
Clear[CCo, to, h, howmany]
to = 0.;
CCo = 10^7;
h = 3.5;
howmany = (7 - to) / h;

Clear[f, CC];
f = -0.06 CC;

Clear[ k1, k2, k3, k4];
Do[
 k1 = f /. {t → to , CC → CCo};
 k2 = f /. {t → to + h / 2 , CC → CCo + k1 h / 2};
 k3 = f /. {t → to + h / 2 , CC → CCo + k2 h / 2};
 k4 = f /. {t → to + h , CC → CCo + k3 h};

 to = to + h;
 CCo = CCo + (k1 + 2 k2 + 2 k3 + k4) h / 6;
 , {k, 1, howmany}]

Clear[cRK];
cRK = CCo
```

$6.57052 \times 10^{6}$

## e) Errors

```
Print["Error between exact and Euler ", 100 Abs[(cExact - cEuler) / cExact] , "%"]

Print["Error between exact and Euler ", 100 Abs[(cExact - cTaylor) / cExact] , "%"]

Print["Error between exact and Euler ", 100 Abs[(cExact - cRK) / cExact] , "%"]
```

Error between exact and Euler 5.01438%

Error between exact and Euler 0.000811214%

Error between exact and Euler 0.000811214%

## The speed of the motor to a voltage input of 20V, assuming a system without damping is

$$20=0.02\,\frac{dw}{dt}+0.06\,w$$

$$w(0)=0$$

a) Make a plot for t<=0<=0.8 s, which includes the following curves:
(i) *Mathematica* result (black, solid);
(ii) Results for step size h=0.8 from Runge-Kutta 4th order method (blue, dashed, and the points);
(iii) Results for step size h=0.4 from Runge-Kutta 4th order method (green, solid, and the points);
(iv) Results for step size h=0.2 from Runge-Kutta 4th order method (red, solid, and the points).
Label the axes.

### (i) *Mathematica*

```
Clear[w, t, sol, wExact, pE];
sol = DSolve[{20 == 0.02 w'[t] + 0.06 w[t], w[0] == 0.}, w[t], t];
wExact = sol[[1, 1, 2]];
pE = Plot[wExact, {t, 0, 0.8}, AxesOrigin → {0, 0}, AxesLabel → {"t", "w"},
   PlotStyle → {Thick, Black}, LabelStyle → Directive[Black, Bold, Medium]];
```

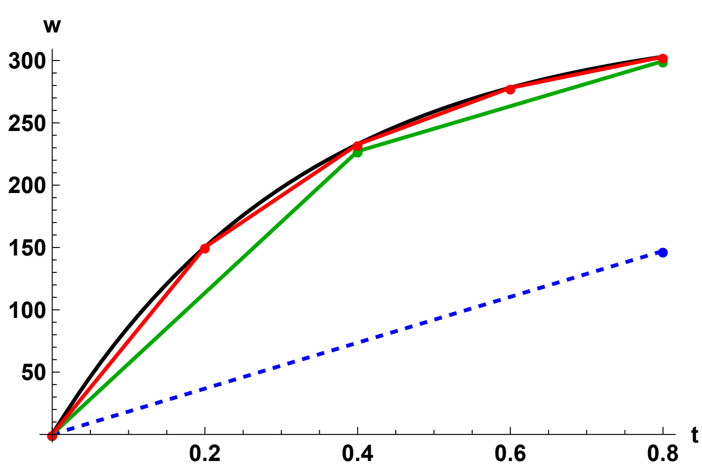
## (ii) Runge - Kutta (h=0.8)

```
Clear[wo, to, W8, h, howmany]
to = 0.;
wo = 0.;
W8[0] = wo;
h = 0.8;
howmany = (0.8 - to) / h;

Clear[f];
f = (20 - 0.06 w) / 0.02;

Clear[ k1, k2, k3, k4];
Do[
 k1 = f /. {t → to , w → wo};
 k2 = f /. {t → to + h / 2 , w → wo + k1 h / 2};
 k3 = f /. {t → to + h / 2 , w → wo + k2 h / 2};
 k4 = f /. {t → to + h , w → wo + k3 h};

 to = to + h;
 wo = wo + (k1 + 2 k2 + 2 k3 + k4) h / 6;
 W8[k] = wo;
 , {k, 1, howmany}]

Clear[l8, p8];
l8 = Table[{h k, W8[k]}, {k, 0, howmany}];
p8 = ListPlot[l8, Joined → True, AxesOrigin → {0, 0},
   AxesLabel → {"t", "w"}, PlotMarkers → Automatic,
   PlotStyle → {Dashed, Thick, Blue}, LabelStyle → Directive[Black, Bold, Medium]];
```

## (ii) Runge - Kutta (h=0.8)

```
Clear[wo, to, W4, h, howmany]
to = 0.;
wo = 0.;
W4[0] = wo;
h = 0.4;
howmany = (0.8 - to) / h;

Clear[f];
f = (20 - 0.06 w) / 0.02;

Clear[ k1, k2, k3, k4];
Do[
 k1 = f /. {t → to , w → wo};
 k2 = f /. {t → to + h / 2 , w → wo + k1 h / 2};
 k3 = f /. {t → to + h / 2 , w → wo + k2 h / 2};
 k4 = f /. {t → to + h , w → wo + k3 h};

 to = to + h;
 wo = wo + (k1 + 2 k2 + 2 k3 + k4) h / 6;
 W4[k] = wo;
 , {k, 1, howmany}]

Clear[l4, p4];
l4 = Table[{h k, W4[k]}, {k, 0, howmany}];
p4 = ListPlot[l4, Joined → True, AxesOrigin → {0, 0},
   AxesLabel → {"t", "w"}, PlotMarkers → Automatic,
   PlotStyle → {Thick, Darker[Green]}, LabelStyle → Directive[Black, Bold, Medium]];
```

```
Clear[wo, to, W2, h, howmany]
to = 0.;
wo = 0.;
W2[0] = wo;
h = 0.2;
howmany = (0.8 - to) / h;

Clear[f];
f = (20 - 0.06 w) / 0.02;

Clear[ k1, k2, k3, k4];
Do[
 k1 = f /. {t → to , w → wo};
 k2 = f /. {t → to + h / 2 , w → wo + k1 h / 2};
 k3 = f /. {t → to + h / 2 , w → wo + k2 h / 2};
 k4 = f /. {t → to + h , w → wo + k3 h};

 to = to + h;
 wo = wo + (k1 + 2 k2 + 2 k3 + k4) h / 6;
 W2[k] = wo;
 , {k, 1, howmany}]

Clear[l2, p2];
l2 = Table[{h k, W2[k]}, {k, 0, howmany}];
p2 = ListPlot[l2, Joined → True, AxesOrigin → {0, 0},
   AxesLabel → {"t", "w"}, PlotStyle → {Thick, Red},
   PlotMarkers → Automatic, LabelStyle → Directive[Black, Bold, Medium]];

Show[{pE, p8, p4, p2}]
```




A solid steel shaft at room temperature of 27$^{o}$C is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at - 33$^o$C. The rate of change of temperature of the solid shaft $\theta$ is given by

$$\frac{d\theta}{dt} = -5.33 \times 10^{-6} \times (-3.69 \times 10^{-6} \theta^4 + 2.33 \times 10^{-5} \theta^3 + 1.35 \times 10^{-3} \theta^2 + 5.42 \times 10^{-2} \theta + 5.588)\,(\theta + 33)$$

$$\theta(0)=27^o\text{C}$$

For some reason *Mathematica* cannot solve this equation!

a) Using Runge-Kutta 4th order method and step size of h = 43200 s , find the temperature of the steel shaft after 86400 s..

b) The same as above but for h = 21600 s.

c) Make a plot of the temperature vs time for 0<=t<=86400.
Show the curves for step size = 14400s, 17280s, 21600s and 28800s.
[hint: check the relationship of these intervals with 86400]
Label the axes and identify the curves with legends.
Use ranges: 0<=x<=90000 and -30<=y<=30.
Use ImageSize→ 500

## a) Runge - Kutta (h=43200 s)

```
Clear[xo, to, h, howmany]
to = 0.;
xo = 27.;
h = 43 200;
howmany = (86 400 - to) / h;

Clear[f, x];
f = -5.33 × 10^(-6) × (-3.69 × 10^(-6) x^4 + 2.33 × 10^(-5) x^3 +
     1.35 × 10^(-3) x^2 + 5.42 × 10^(-2) x + 5.588) (x + 33);

Clear[ k1, k2, k3, k4];
Do[
 k1 = f /. {t → to , x → xo};
 k2 = f /. {t → to + h / 2 , x → xo + k1 h / 2};
 k3 = f /. {t → to + h / 2 , x → xo + k2 h / 2};
 k4 = f /. {t → to + h , x → xo + k3 h};

 to = to + h;
 xo = xo + (k1 + 2 k2 + 2 k3 + k4) h / 6;
 , {k, 1, howmany}]

Clear[xRK];
xRK = xo
```

$-1.01049 \times 10^{23}$

## b) Runge - Kutta (h=21600 s)

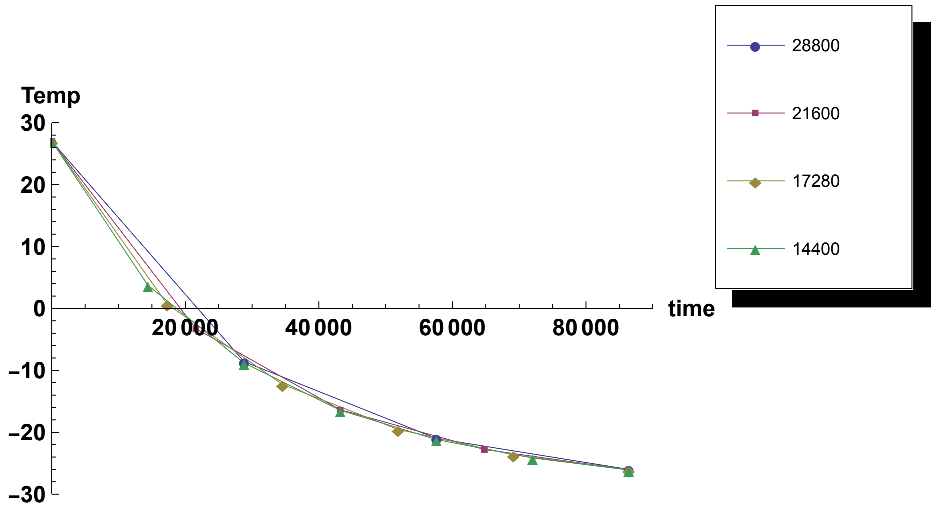
```
Clear[xo, to, h, howmany];
to = 0.;
xo = 27.;
h = 21 600;
howmany = (86 400 - to) / h;

Clear[XX, TT];
TT[0] = to;
XX[0] = xo;

Clear[f, x];
f = -5.33 × 10^(-6) × (-3.69 × 10^(-6) x^4 + 2.33 × 10^(-5) x^3 +
      1.35 × 10^(-3) x^2 + 5.42 × 10^(-2) x + 5.588) (x + 33);

Clear[ k1, k2, k3, k4];
Do[
  k1 = f /. {t → to , x → xo};
  k2 = f /. {t → to + h / 2 , x → xo + k1 h / 2};
  k3 = f /. {t → to + h / 2 , x → xo + k2 h / 2};
  k4 = f /. {t → to + h , x → xo + k3 h};

  to = to + h;
  xo = xo + (k1 + 2 k2 + 2 k3 + k4) h / 6;

  TT[k] = to;
  XX[k] = xo;

  , {k, 1, howmany}];

Clear[lis1];
lis1 = Table[{TT[k], XX[k]}, {k, 0, howmany}];

Clear[xRK];
xRK = xo
```

```
-26.0603
```

## c) Runge - Kutta (PLOT)

```
<< PlotLegends`;
```

```
Clear[lis, final];
final = 86 400;

Do[

  Clear[xo, to, h, howmany];
  to = 0.;
  xo = 27.;
  h = final / kk;
  howmany = (final - to) / h;

  Clear[XX, TT];
  TT[0] = to;
  XX[0] = xo;

  Clear[f, x];
  f = -5.33 × 10^(-6) × (-3.69 × 10^(-6) x^4 + 2.33 × 10^(-5) x^3 +
      1.35 × 10^(-3) x^2 + 5.42 × 10^(-2) x + 5.588) (x + 33);

  Clear[ k1, k2, k3, k4];
  Do[
   k1 = f /. {t → to , x → xo};
   k2 = f /. {t → to + h / 2 , x → xo + k1 h / 2};
   k3 = f /. {t → to + h / 2 , x → xo + k2 h / 2};
   k4 = f /. {t → to + h , x → xo + k3 h};

   to = to + h;
   xo = xo + (k1 + 2 k2 + 2 k3 + k4) h / 6;

   TT[k] = to;
   XX[k] = xo;

   , {k, 1, howmany}];

  lis[kk - 2] = Table[{TT[k], XX[k]}, {k, 0, howmany}];
  , {kk, 3, 6}];

ListPlot[Table[lis[k], {k, 1, 4}], Joined → True, AxesLabel → {"time", "Temp"},
 PlotLegend → {"28800", "21600", "17280", "14400"}, PlotMarkers → Automatic,
 LegendPosition → {1, 0}, LabelStyle → Directive[Black, Bold, Medium],
 PlotRange → {{0, 90 000}, {-30, 30}}, ImageSize → 500]
```




QUESTIONS for written assignment and presentations

What are soaps made of? What are they for? How does a handmade soap maker work?

What are the types of contamination waters in rivers and lakes face? What are the methods available to clean them?

What are steel shafts used for? What are hollow hubs? What is steel? How is it produced?