Monte Carlo methods

*) How to approximate the value of Pi.
We know that the area of a circle is \( \pi R^2 \).
Pick a circle of radius \( R=1 \) and surround it with a square of side 1.
Select just quadrant where the possible values of \( x \) and \( y \) in the square can vary from 0 to 1.
Get various random numbers in this range for both \( x \) and \( y \).
Use Pythagoras to find the hypotenuse of the respective triangles.
If this value is \( \leq 1 \), it belongs to the \( 1/4 \) of the circle.
We sum all the points in this condition. The result is proportional to area of the \( 1/4 \) of the circle.
We have:
\[
\frac{\text{Number of points in } 1/4 \text{ of circle}}{\text{Total number of points}} = \frac{\text{area of } 1/4 \text{ of circle}}{\text{area of square}} = \frac{\pi R^2/4}{R^2}
\]

Clear[shade, tot, sideX, sideY, hypo];
shade = 0;
tot = 10000;
Do[
    sideX = RandomReal[];
    sideY = RandomReal[];
    hypo = Sqrt[sideX^2 + sideY^2];
    If[hypo \leq 1., shade = shade + 1];
, {k, 1, tot}];

Print["From Monte Carlo we get Pi=", 4. shade / tot];
Print["The actual value is ", 1. Pi]

From Monte Carlo we get Pi=3.1572
The actual value is 3.14159
*) Find 2000 values of Pi using the method above.
What is the average and the variance?
Show a histogram with the distribution of those values.
Clear[tot, Nrea];
Nrea = 2000;
tot = 10000;
Do[
  Clear[shade, sideX, sideY, hypo];
  shade = 0;
  Do[
    sideX = RandomReal[];
    sideY = RandomReal[];
    hypo = Sqrt[sideX^2 + sideY^2];
    If[hypo ≤ 1., shade = shade + 1];
    , {k, 1, tot}];
  pip[kk] = 4. shade / tot;
  , {kk, 1, Nrea}];

Clear[la];
la = Table[pip[kk], {kk, 1, Nrea}];
Print["Average:"];
Sum[la[[kk]], {kk, 1, Nrea}]/Nrea
Print["Variance:"];
Sum[la[[kk]]^2, {kk, 1, Nrea}]/Nrea - (Sum[la[[kk]], {kk, 1, Nrea}]/Nrea)^2
Histogram[la]

Average:
3.14164

Variance:
0.000276191

*) **Birthday Problem.**
We want to find out the probability that out of 30 people two share a birthday.
Person 1
Person 2: prob=364/365 of no overlap with the first;
Person 3: prob=363/365 of no overlap with 1 and 2;
Person 4: prob=362/365 of no overlap with 1, 2 and 3;
...
Person 30: prob=336/365 of no overlap with any person above;

The probability of having no shared birthdays is then (364/365)*(363/365)*(362/365)...(336/365) =

\[
\text{Product}\left[1. \times \frac{(365-k)}{365}, \{k, 1, 29\}\right]
\]

\[0.293684\]

So the probability of having at least one pair of people having the same birthday is 71%.

Let us find this probability with the Monte Carlo Approach:
1) Pick 30 random numbers in the range [1,365].
2) Check to see if any of the thirty are equal.
3) Go back to step 1 and repeat 10000 times.
4) Report the fraction of trial that have matching birthdays.

Clear[trials, Npeople, matches];
trials = 10000;
Npeople = 30;
matches = 0;

Do[
    Clear[radlis];
    radlis = Table[RandomInteger[{1, 365}], {k, 1, Npeople}];

    Do[
        Do[
            If[radlis[[k]] == radlis[[k + j]],
                matches = matches + 1,
                Goto[end]];
            , {j, 1, Npeople - k}];
        , {k, 1, Npeople - 1}];
    Label[end];
    , {m, 1, trials}];
1. matches / trials

0.7032

\[\int_a^b f(x) \, dx \sim \sum_{i=0}^{n-1} f(x_i) \Delta x\]

\[\Delta x = (b-a)/n\]
Different methods can be used to solve the integral above [see Lesson05 and Test 2]

Monte Carlo:
- Select random numbers between a and b to compute f(x)
- Compute the sum to approximate the integral

Solve the examples below exactly and with Monte Carlo:

(i) \( \int_0^1 x \, dx \)

(ii) \( \int_0^2 x^2 \, dx \)

(iii) \( \int_0^\pi \sin(x) \, dx \)

Clear[Nt];
Nt = 10000;

Print["Item (i)""]
Print["Exact:"];
Integrate[1. x, {x, 0, 1}]
Print["Monte Carlo:"];
Sum[RandomReal[], {k, 1, Nt}] (1./Nt)

Print["Item (ii)""]
Print["Exact:"];
Integrate[1. x^2, {x, 0, 2}]
Print["Monte Carlo:"];
Sum[RandomReal[{0, 2}]^2, {k, 1, Nt}] (2./Nt)

Print["Item (iii)""]
Print["Exact:"];
Integrate[1. Sin[x], {x, 0, Pi}]
Print["Monte Carlo:"];
Sum[Sin[RandomReal[{0, Pi}]], {k, 1, Nt}] (Pi/Nt)

Item (i) Exact:
0.5
Monte Carlo:
0.502353
Item (ii)
Exact:
2.66667
Monte Carlo:
2.67117

Item (iii)
Exact:
2.
Monte Carlo:
1.9979