Lesson 10

Binomial Distribution

EXERCISE 1 (a)
If you take N=3 steps in 1D, what are all the possible sequences of steps?

```
Permutations[{R, R, R}]
Permutations[{R, R, L}]
Permutations[{R, L, L}]
Permutations[{L, L, L}]
```

```
{{R, R, R}}

{{R, R, L}, {R, L, R}, {L, R, R}}

{{R, L, L}, {L, R, L}, {L, L, R}}

{{L, L, L}}
```

NOTE
To get all possible sequences, we can write the total number of steps in base 2.

```
BaseForm[3, 2]
BaseForm[7, 2]
BaseForm[8, 2]
```

11_2

111_2

1000_2

```
IntegerDigits[3, 2]
IntegerDigits[7, 2]
IntegerDigits[8, 2]
```

```
{1, 1}

{1, 1, 1}

{1, 0, 0, 0}
```

```
IntegerDigits[3, 2, 4]
```

```
{0, 0, 1, 1}
```

```
nn = 3;
Do[
 Print[{k, IntegerDigits[k, 2, 3]}];
 , {k, 0, 2^nn - 1}]
```

```
{0, {0, 0, 0}}
{1, {0, 0, 1}}
{2, {0, 1, 0}}
{3, {0, 1, 1}}
{4, {1, 0, 0}}
{5, {1, 0, 1}}
{6, {1, 1, 0}}
{7, {1, 1, 1}}
```

<u>EXERCISE 1 (b)</u>

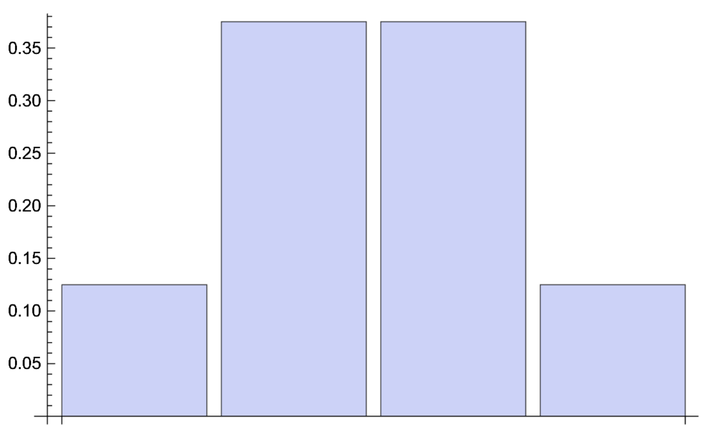
If you take N=3 steps and the probability to go to the right and left are the same p=q=1/2, what are the probabilities of taking nr steps to the right, such that nr=0, 1, ... N?

Make a plot (barchart) with the values of these probabilities.

```
Clear[p, q, nn, prob, tab];
p = 0.5;
q = 0.5;
nn = 3;
prob[x_] := nn! / (x! (nn - x)!) p^x q^(nn - x)
tab = Table[{nr, prob[nr]}, {nr, 0, nn}]
```

```
{{0, 0.125}, {1, 0.375}, {2, 0.375}, {3, 0.125}}
```

```
Clear[teb]
teb = Table[prob[nr], {nr, 0, nn}];
BarChart[teb]
```



<u>EXERCISE 2 (a)</u>

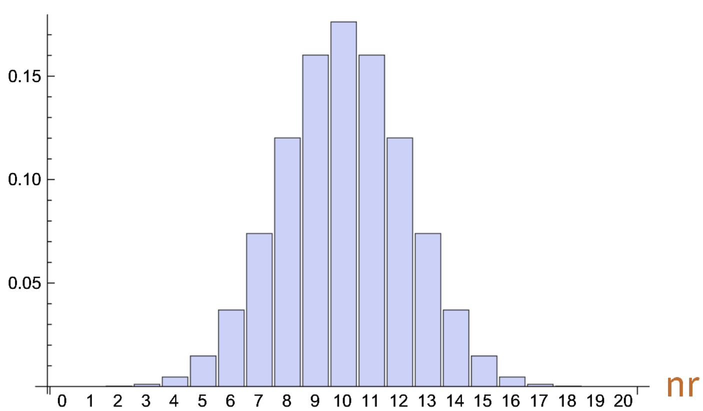
If you take N=20 steps and the probability to go to the right and left are the same p=q=1/2, what are the probabilities of taking nr steps to the right, such that nr=0, 1, ... N?

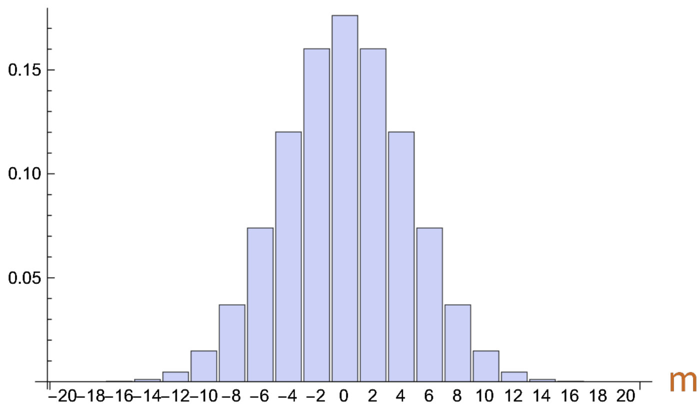
Make a plot (barchart) with the values of these probabilities.
Does the shape of the plot look familiar?

```
Clear[p, q, nn, prob, tab];
p = 0.5;
q = 0.5;
nn = 20;
prob[x_] := nn! / (x! (nn - x)!) p^x q^(nn - x)
tab = Table[{nr, prob[nr]}, {nr, 0, nn}]

Clear[teb]
teb = Table[prob[nr], {nr, 0, nn}];
BarChart[teb, ChartLabels → Table[nr, {nr, 0, nn}], AxesLabel → {"nr", ""}]
BarChart[teb, ChartLabels → Table[2 nr - nn, {nr, 0, nn}], AxesLabel → {"m", ""}]
```

{{0, 9.53674 × 10^-7}, {1, 0.0000190735}, {2, 0.000181198}, {3, 0.00108719}, {4, 0.00462055}, {5, 0.0147858}, {6, 0.0369644}, {7, 0.0739288}, {8, 0.120134}, {9, 0.160179}, {10, 0.176197}, {11, 0.160179}, {12, 0.120134}, {13, 0.0739288}, {14, 0.0369644}, {15, 0.0147858}, {16, 0.00462055}, {17, 0.00108719}, {18, 0.000181198}, {19, 0.0000190735}, {20, 9.53674 × 10^-7}}





EXERCISE 2 (b)
(i) What is the sum of the probabilities for all values of nr?
(ii) What is the average for nr and m?
(iii) What is the variance for nr and m?

```
Print["Item (i): summ of probabilities"];
Sum[prob[x], {x, 0, nn}]
Print["Item (ii): average <nr> and <m>"];
Sum[prob[x] x, {x, 0, nn}]
Sum[prob[x] (2 x - nn), {x, 0, nn}]
Print["Item (iii): variance <nr^2>-<nr>^2 and for m"];
Sum[prob[x] x^2, {x, 0, nn}] - (Sum[prob[x] x, {x, 0, nn}])^2
Sum[prob[x] (2 x - nn)^2, {x, 0, nn}] - (Sum[prob[x] (2 x - nn), {x, 0, nn}])^2
```

```
Item (i): summ of probabilities

1.

Item (ii): average <nr> and <m>

10.

0.

Item (iii): variance <nr^2>-<nr>^2 and for m

5.

20.
```

EXERCISE 2 (c)
Repeat the questions:
(ii) What is the average for nr and m?
(iii) What is the variance for nr and m?
For N=10, 20, 30, 40, ...100. Can you generalize your results for <nr>, <m> and the variances in terms of N?

```
Clear[p, q, nn, prob];
p = 0.5;
q = 0.5;
prob[x_, Nt_] := Nt! / (x! (Nt - x)!) p^x q^(Nt - x)

Do[
 Clear[nn, ave, avem, var, varm];
 nn = 10 k;
 ave = Sum[prob[nr, nn] nr, {nr, 0, nn}];
 avem = Sum[prob[nr, nn] (2 nr - nn), {nr, 0, nn}];
 var = Sum[prob[nr, nn] nr^2, {nr, 0, nn}] - Sum[prob[nr, nn] nr, {nr, 0, nn}]^2;
 varm = Sum[prob[nr, nn] (2 nr - nn)^2, {nr, 0, nn}] -
   Sum[prob[nr, nn] (2 nr - nn), {nr, 0, nn}]^2;
 Print[{nn, ave, Chop[avem], var, varm}];
 , {k, 1, 10}]
```

```
{10, 5., 0, 2.5, 10.}
{20, 10., 0, 5., 20.}
{30, 15., 0, 7.5, 30.}
{40, 20., 0, 10., 40.}
{50, 25., 0, 12.5, 50.}
{60, 30., 0, 15., 60.}
{70, 35., 0, 17.5, 70.}
{80, 40., 0, 20., 80.}
{90, 45., 0, 22.5, 90.}
{100, 50., 0, 25., 100.}
```

Conclusion:

<nr>=N/2 and variance = N/4

<m>=0 and variance = N

EXERCISE 2 (d)

Repeat the problem above, but for p=0.1 and q=0.9.
Can you generalize your results for <nr>, <m> and the variances in terms of N, p and q?

```
Clear[p, q, nn, prob];
p = 0.1;
q = 0.9;
prob[x_, Nt_] := Nt! / (x! (Nt - x)!) p^x q^(Nt - x)

Do[
 Clear[nn, ave, avem, var, varm];
 nn = 10 k;
 ave = Sum[prob[nr, nn] nr, {nr, 0, nn}];
 avem = Sum[prob[nr, nn] (2 nr - nn), {nr, 0, nn}];
 var = Sum[prob[nr, nn] nr^2, {nr, 0, nn}] - Sum[prob[nr, nn] nr, {nr, 0, nn}]^2;
 varm = Sum[prob[nr, nn] (2 nr - nn)^2, {nr, 0, nn}] -
   Sum[prob[nr, nn] (2 nr - nn), {nr, 0, nn}]^2;
 Print[{nn, ave, Chop[avem], var, varm}];
 , {k, 1, 10}]
```

```
{10, 1., -8., 0.9, 3.6}
{20, 2., -16., 1.8, 7.2}
{30, 3., -24., 2.7, 10.8}
{40, 4., -32., 3.6, 14.4}
{50, 5., -40., 4.5, 18.}
{60, 6., -48., 5.4, 21.6}
{70, 7., -56., 6.3, 25.2}
{80, 8., -64., 7.2, 28.8}
{90, 9., -72., 8.1, 32.4}
{100, 10., -80., 9., 36.}
```

Conclusion:

<nr>=Np and the variance = Npq
<m>=N(p-q) and the variance = 4Npq

Let us now obtain these results analytically.

Random Walk in 1D

EXERCISE 3

(i) One realization of a random walk with 10 steps:
Make a list/table with 10 random numbers which can only be +1 or -1. Give a name to this list.

```
rw = Table[(-1)^RandomInteger[], {k, 1, 10}]
{1, -1, 1, -1, 1, -1, 1, -1, 1, 1}
```

(ii) What is the final position of the particle after those 10 steps?

```
Sum[rw[[k]], {k, 1, 10}]
```

```
2
```

(iii) Assume N=100 and 200 realizations of random walks.
Compute the average final position and its variance. Do the results agree with your expectations? What is the relative error for the variance?
Make a histogram with the final positions.

```
Clear[Nt, Nrea];
Nt = 100;
Nrea = 200;
Clear[rw, FinPos];
Do[
  rw = Table[(-1)^RandomInteger[], {k, 1, Nt}];
  FinPos[j] = Sum[rw[[k]], {k, 1, Nt}];
  , {j, 1, Nrea}];

Clear[ave, var];
Print["Average"];
ave = Sum[1. FinPos[j], {j, 1, Nrea}] / Nrea
Print["Variance"];
var = Sum[1. FinPos[j]^2, {j, 1, Nrea}] / Nrea -
  (Sum[1. FinPos[j], {j, 1, Nrea}] / Nrea)^2
Print["The relative error is ", 100 Abs[var - Nt] / Nt, "%"]

Print["Histogram"];
Clear[tab];
tab = Table[FinPos[j], {j, 1, Nrea}];
Histogram[tab]
```

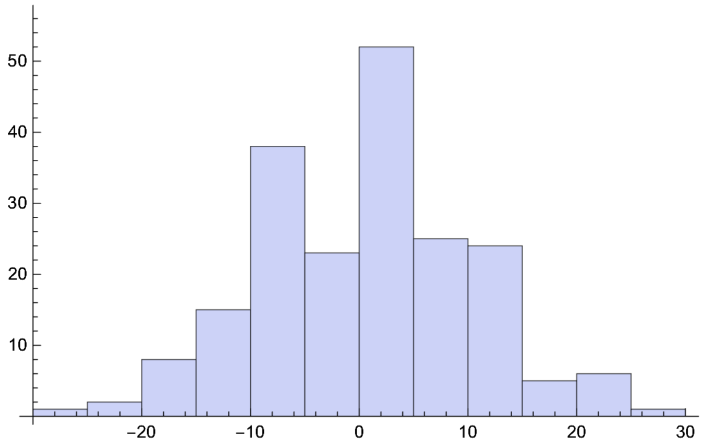
```
Average

0.07

Variance

97.5351
```

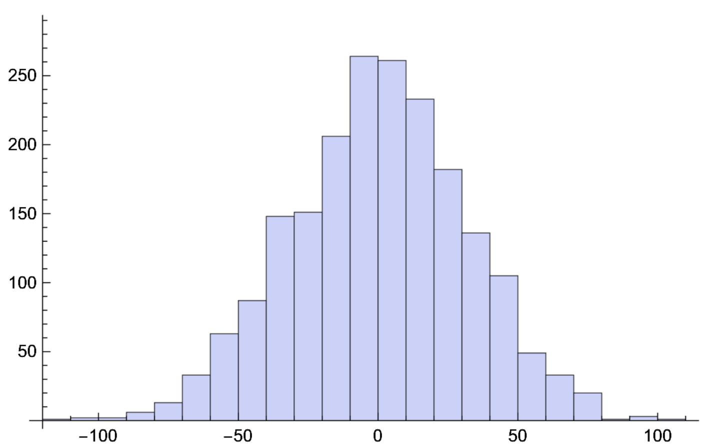
```
The relative error is 2.4649%
Histogram
```



(iv) Assume N=1000 and 2000 realizations of random walks.
Compute the average final position and its variance. Do the results agree with your expectations? What is the relative error for the variance?
Make a histogram with the final positions.

```
Clear[Nt, Nrea];
Nt = 1000;
Nrea = 2000;
Clear[rw, FinPos];
Do[
  rw = Table[(-1)^RandomInteger[], {k, 1, Nt}];
  FinPos[j] = Sum[rw⟦k⟧, {k, 1, Nt}];
  , {j, 1, Nrea}];
Clear[ave, var];
Print["Average"];
ave = Sum[1. FinPos[j], {j, 1, Nrea}] / Nrea
Print["Variance"];
var = Sum[1. FinPos[j]^2, {j, 1, Nrea}] / Nrea -
  (Sum[1. FinPos[j], {j, 1, Nrea}] / Nrea)^2
Print["The relative error is ", 100 Abs[var - Nt] / Nt, "%"]
Print["Histogram"];
Clear[tab];
tab = Table[FinPos[j], {j, 1, Nrea}];
Histogram[tab]
```

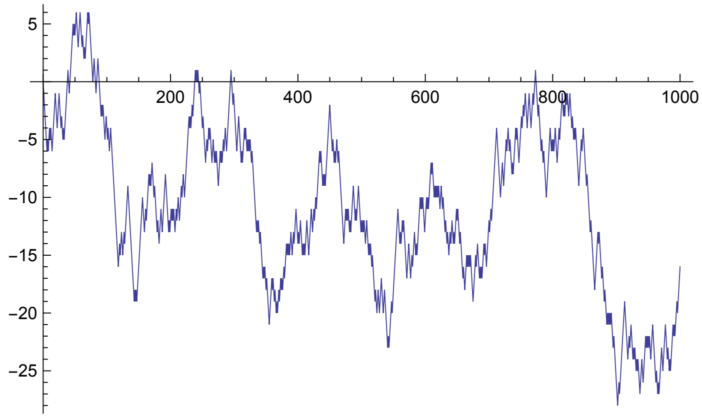
```
Average
-0.553
Variance
1001.12
The relative error is 0.112419%
Histogram
```

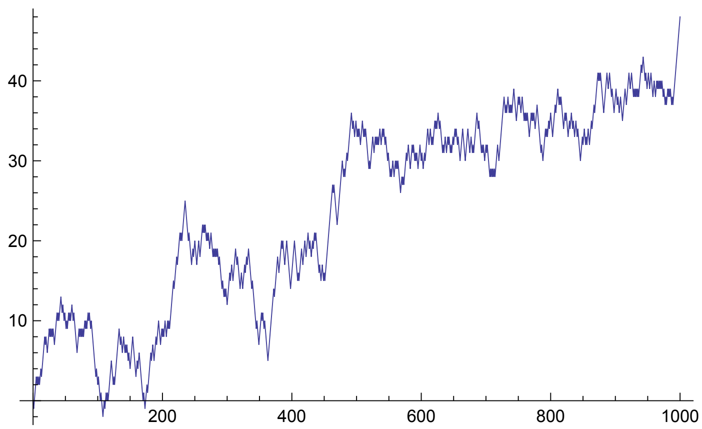


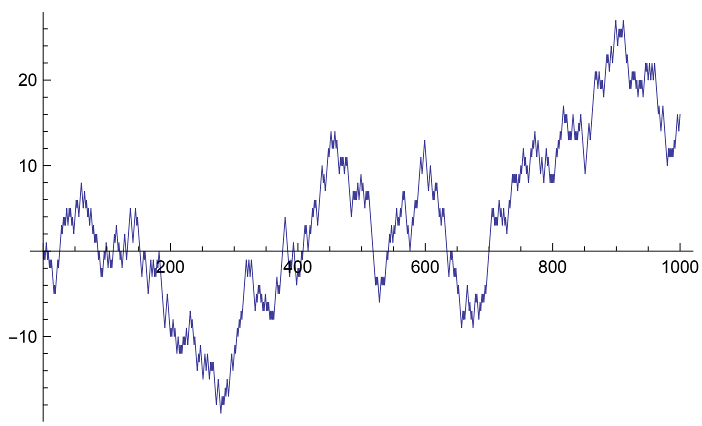
(v)
*) Get a sequence of random steps in a 1D Random Walk with N=1000 steps.
*) Get the position of the particle after every step. Start with position zero, and then add the steps of the sequence above for 1 step, 2 steps, 3 steps,... N steps.

*) Make a plot of Number of Steps in the x-axis and Position in the y-axis.
*) Repeat it 3 times.

```
Clear[Nt];
Nt = 1000;
Do[
 Clear[rw, tabSum, init, pos];
 rw = Table[(-1)^RandomInteger[], {k, 1, Nt}];
 tabSum = Table[{j, Sum[rw[[k]], {k, 1, j}]}, {j, 1, Nt}];
 init = {0, 0};
 pos = Prepend[tabSum, init];
 Print[ListPlot[pos, Joined → True]];
 , {k, 1, 3}]
```







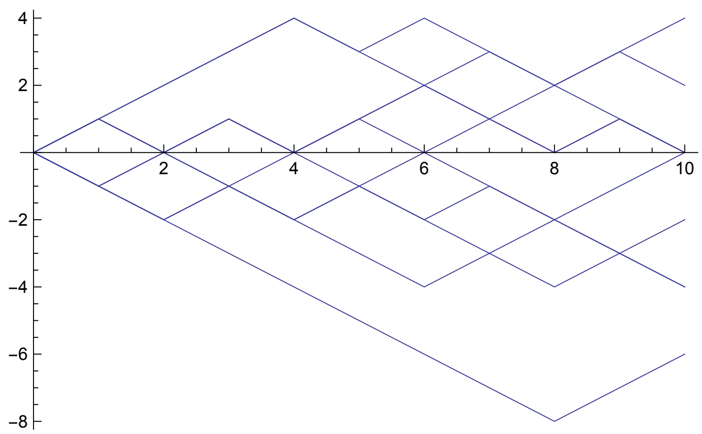
(vi)

*) Use now only N=10 steps and plot together 10 different realizations

*) Repeat it for 300 realizations

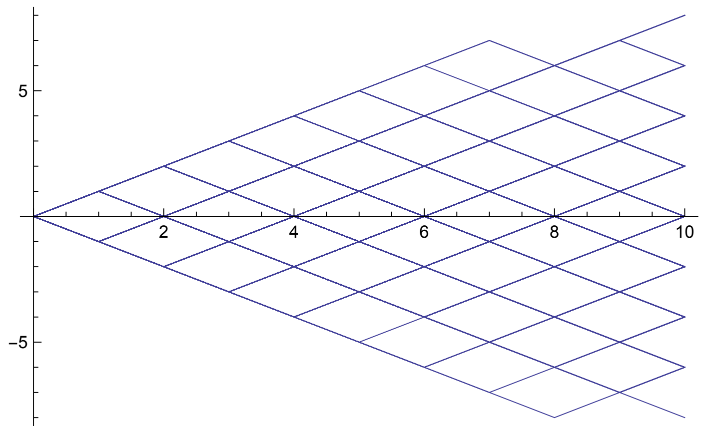
```
Clear[Nrea, Nt, la, pos];
Nrea = 10;
Nt = 10;
Do[
  Clear[rw, tabSum, pos];
  rw = Table[(-1)^RandomInteger[], {k, 1, Nt}];
  tabSum = Table[{j, Sum[rw[[k]], {k, 1, j}]}, {j, 1, Nt}];
  init = {0, 0};
  pos = Prepend[tabSum, init];
  la[k] = ListPlot[pos, Joined → True];
  , {k, 1, Nrea}];

Show[Table[la[k], {k, 1, Nrea}], PlotRange → All]
```



```
Clear[Nrea, Nt, la, pos];
Nrea = 300;
Nt = 10;
Do[
  Clear[rw, tabSum, pos];
  rw = Table[(-1)^RandomInteger[], {k, 1, Nt}];
  tabSum = Table[{j, Sum[rw⟦k⟧, {k, 1, j}]}, {j, 1, Nt}];
  init = {{{0, 0}}};
  pos = Flatten[AppendTo[init, tabSum], 1];
  la[k] = ListPlot[pos, Joined → True];
  , {k, 1, Nrea}];

Show[Table[la[k], {k, 1, Nrea}], PlotRange → All]
```



Gaussian Distribution

$$P(x)=\frac{1}{\sqrt{2\,\pi\sigma^2}}\exp(\frac{-(x-\mu)^2}{2\,\sigma^2})$$

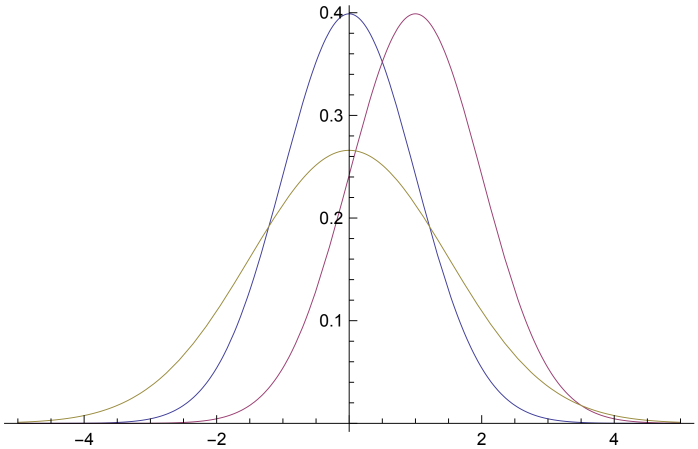
(a) is normalized;

(b) <x> = μ;

(c) variance = σ^2

*) Plot three Gaussians: (i) mu=0, sigma=1; (ii) mu=1, sigma=1; (iii) mu=0, sigma=1.5
*) Select one of them and show that it is normalized

```
Clear[mu1, mu2, sig1, sig2];
mu1 = 0;
mu2 = 1;
sig1 = 1;
sig2 = 1.5;
g1 = 1 / Sqrt[2 Pi sig1^2] Exp[- (x - mu1)^2 / (2 sig1^2)];
g2 = 1 / Sqrt[2 Pi sig1^2] Exp[- (x - mu2)^2 / (2 sig1^2)];
g3 = 1 / Sqrt[2 Pi sig2^2] Exp[- (x - mu1)^2 / (2 sig2^2)];
Plot[{g1, g2, g3}, {x, -5, 5}]
```




```
Integrate[g1, {x, -Infinity, Infinity}]
```

```
1
```

Diffusion

$$\rho(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(\frac{-x^2}{4 D t}\right)$$

is a solution of the diffusion equation

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$