SPIN

In classical mechanics

rigid object has

→ <u>orbital</u> angular momentum

$\vec{L} = \vec{r} \times \vec{p}$

(Earth around sun)

→ <u>spin</u> angular momentum

$\vec{S} = I\vec{\omega}$

(Earth daily rotation)

(can be decomposed into orbital angular momenta of constituent parts)

<u>By analogy</u>

In quantum mechanics

→ <u>orbital</u> angular momentum (extrinsic angular momentum)

Ex. e^- around the nucleus in H atom

(spherical harmonics)

→ $\boxed{\text{spin}}$ – nothing to do with motion in space

not $\cancel{f(\theta, \phi, r)}$

e^- is <u>structureless</u> point particle

= <u>INTRINSIC</u> angular momentum $\vec{S}$

<u>Algebraic</u> theory of spin follows that of orbital angular momentum

(formulas derived from rotational invariance in 3D)

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$\left\{ \begin{array}{l} S^2 |sm\rangle = \hbar^2 s(s+1) |sm\rangle \\ S_z |sm\rangle = \hbar m |sm\rangle \end{array} \right\}$$

<u>Ket</u> notation, because eigenstates of spin are <u>NOT</u> <u>functions</u>

[For $\vec{L}$, we can use $|\ell m\rangle$ (Ket) or $Y_\ell^m(\theta, \phi)$ (function)]

As in Prob. 4.18

$$L_\pm f_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}\; f_\ell^{m\pm1}$$

$\left\{ (L_\pm)^\dagger = L_\mp \right.$
because L_x, L_y are hermitian

$$L_\pm f_\ell^m = A_\ell^m f_\ell^{m\pm1}$$

$$\langle f_\ell^m | L_\mp L_\pm | f_\ell^m\rangle = \langle f_\ell^m | L^2 - L_z^2 \mp \hbar L_z | f_\ell^m\rangle = \langle f_\ell^m | \hbar^2 \ell(\ell+1) - \hbar^2 m^2 \mp \hbar^2 m | f_\ell^m\rangle$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\langle L_\pm f_\ell^m | L_\pm f_\ell^m\rangle = \underbrace{|A_\ell^m|^2 \qquad\qquad}\quad \hbar^2\ell(\ell+1) - \hbar^2 m(m\pm1)$$

$$\boxed{A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}}$$

$$\boxed{S_\pm |s\,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\;|s\,(m\pm1)\rangle}$$

Comment : ◦) for orbital angular momentum (ℓ) is a nonnegative integer so that Eq. (4.28) with ($\ell!$) makes sense

However, from the algebraic method we saw that (m) goes from ($-\ell$) to (ℓ) in (N) integer steps $\Rightarrow -\ell + N = \ell$

$\Rightarrow \underbrace{\ell = N/2}$

which says that (ℓ) may be an $\boxed{\text{integer}}$ or a $\cancel{\boxed{\text{half-integer}}}$

contradicts separation of variables

◦) for spin angular momentum

then (s) can indeed be integer or half-integer

$\left\{\begin{array}{l} \pi \text{ meson}: s=0 \\ e^-: s = 1/2 \\ \text{photon}: s=0 \\ \text{graviton}: s=2 \end{array}\right.$

$$\boxed{s = 0, 1/2, 1, 3/2, \ldots} \qquad \boxed{m = -s, -s+1, \ldots s-1, s}$$

Every elementary particle has a specific and fixed spin

(while ℓ can change)

SPIN 1/2

↳ particles that make ordinary matter
protons, neutrons, electrons
quarks, leptons (e^-, neutrinos, muons)

$s = 1/2 \Rightarrow m = -1/2$ or $+1/2 \Rightarrow$ there are only <u>two</u> eigenstates of S_z

$$\begin{cases} \boxed{\text{spin up}} \ |\uparrow\rangle \quad \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle \longrightarrow \chi_+^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ \boxed{\text{spin down}} \ |\downarrow\rangle \quad \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \longrightarrow \chi_-^{(z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

- Using these two states as basis vectors, we can write any general state of a spin-1/2 particle as a <u>two-element</u> $\underbrace{\text{column matrix}}_{\text{vector}}$ (<u>spinor</u>)

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\,\chi_+ + b\,\chi_- \qquad \Big\{ \text{general state}$$

notation $\chi_\pm$ or $\chi_\pm^z$ to remember that these are eigenstates of S_z

- spin operators: <u>2x2 matrices</u>

$$\begin{cases} S^2 \chi_+ = \hbar^2 s(s+1)\,\chi_+ = \frac{3}{4}\hbar^2 \chi_+ \\ S^2 \chi_- = \frac{3}{4}\hbar^2 \chi_- \end{cases}$$

[assuming $\chi_\pm$ are eigenstates of S^2 besides S_z (analogy with L)]

$$S^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$$

$$S^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ e \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c = 3/4\,\hbar^2 \\ e = 0 \end{cases}$$

$$S^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} d \\ f \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} d = 0 \\ f = 3/4\,\hbar^2 \end{cases}$$

$$\boxed{S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\left\{ \begin{aligned} S_z \chi_+ &= \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ S_z \chi_- &= \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \right\} \quad \boxed{S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$S_\pm |s\, m\rangle = \hbar \sqrt{s(s+1) - m(m\pm 1)}\; |s\ (m\pm 1)\rangle$$

$$\left\{ \begin{aligned} & S_+ \chi_+ = 0 \qquad S_- \chi_- = 0 \\ & S_+ \chi_- = \hbar \sqrt{1/2\,(3/2) - (-1/2)(-1/2 + 2/2)}\; \chi_+ = \hbar \chi_+ \\ & S_- \chi_+ = \hbar \sqrt{1/2\,(3/2) - (1/2)(1/2 - 2/2)}\; \chi_- = \hbar \chi_- \end{aligned} \right.$$

S_+

$$\left\{ \begin{aligned} & \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c = 0 \\ e = 0 \end{cases} \\ & \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} d = \hbar \\ f = 0 \end{cases} \end{aligned} \right.$$

S_-

$$\left\{ \begin{aligned} & \begin{pmatrix} C & D \\ E & F \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} D = 0 \\ F = 0 \end{cases} \\ & \begin{pmatrix} C & D \\ E & F \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} C = 0 \\ E = \hbar \end{cases} \end{aligned} \right.$$

$$\boxed{S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \qquad \boxed{S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

Since $S_\pm = S_x \pm i S_y \Rightarrow S_x = (S_+ + S_-)/2$ and $S_y = (S_+ - S_-)/(2i)$

$$\boxed{S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \qquad \boxed{S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

∘) We can write $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

where

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are PAULI MATRICES

∘) Note: S_x, S_y, S_z, S^2 are hermitian (they represent observables)
but S_+ and S_- are NOT

∘) The eigenspinors of S_z are $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

eigenvalue $+\hbar/2$ eigenvalue $-\hbar/2$

so if we measure S_z
on a particle in a general state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

we get $+\hbar/2$ with probability $\boxed{|a|^2}$
and $-\hbar/2$ with probability $\boxed{|b|^2}$

where $|a|^2 + |b|^2 = 1$
because (spinor is NORMALIZED)

∘) What happens if we measure S_x ?

First We need to write the general state χ in the basis vectors consisting of the eigenvectors of S_x
(eigenspinors)

Eigenvalues and eigenvectors of $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

characteristic equation

$$\begin{vmatrix}-\lambda & \hbar/2 \\ \hbar/2 & -\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}$$

$$\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow \begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow \boxed{\beta = \pm\alpha}$$

after normalizing

$$\chi_+^{(x)} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} \qquad \chi_-^{(x)} = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$$

eigenvalue $+\hbar/2$ eigenvalue $-\hbar/2$

Therefore the general state

$$\chi = \begin{pmatrix}a\\ b\end{pmatrix} = \boxed{\left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)}}$$

If you measure $S_x \Rightarrow \begin{cases} \frac{|a+b|^2}{2} & \text{is the prob. of getting } +\hbar/2 \\ \frac{|a-b|^2}{2} & \text{"} \qquad -\hbar/2 \end{cases}$

Example 4.2 $\chi = \frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\ 2\end{pmatrix}$ What are the probs. for $+\hbar/2$ and $-\hbar/2$ if you measure S_z and S_x? compute $\langle S_x\rangle$.

$$S_z \begin{cases} +\hbar/2 \Rightarrow |1+i|^2/6 = \boxed{1/3} \\ -\hbar/2 \Rightarrow 4/6 = \boxed{2/3} \end{cases}$$

$$S_x \begin{cases} +\hbar/2 \Rightarrow |3+i|^2/(6.2) = \boxed{5/6} \\ -\hbar/2 \Rightarrow |-1+i|^2/(6.2) = \boxed{1/6} \end{cases} \Rightarrow \langle S_x\rangle = \frac{5}{6}\frac{\hbar}{2} - \frac{1}{6}\frac{\hbar}{2} = \boxed{\hbar/3}$$