•) By the algebraic method we found the eigenvalues of L^2 and L_z

•) We now ~~construct~~ find the eigenfunctions
and show that they are the spherical harmonics

$$f_\ell^m = Y_\ell^m$$

•) L^2 and L_z are hermitian

↳ Y_ℓ^m are orthogonal (for ≠ eigenvalues)

$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i} \vec{r} \times \vec{\nabla}$$

(extra material) →
$$\left\{ \begin{array}{l} \vec{\nabla} = \hat{r} \dfrac{\partial}{\partial r} + \hat{\theta} \dfrac{1}{r} \dfrac{\partial}{\partial \theta} + \hat{\phi} \dfrac{1}{r \sin\theta} \dfrac{\partial}{\partial \phi} \\ \vec{r} = r \hat{r} \end{array} \right.$$

$\hat{r}, \hat{\theta}, \hat{\phi}$ } unit vectors

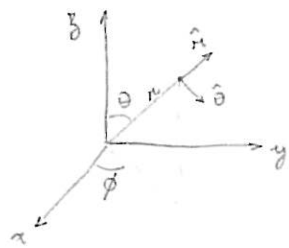
$$\vec{L} = \frac{\hbar}{i} \left[r \cancel{(\hat{r} \times \hat{r})} \frac{\partial}{\partial r} + (\hat{r} \times \hat{\theta}) \frac{\partial}{\partial \theta} + (\hat{r} \times \hat{\phi}) \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right]$$

$$\hat{r} \times \hat{\theta} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\phi} \qquad \hat{r} \times \hat{\phi} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\hat{\theta}$$

$$\boxed{\vec{L} = \frac{\hbar}{i} \left[\hat{\phi} \frac{\partial}{\partial \theta} - \hat{\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right]}$$

since (extra material) →
$$\left\{ \begin{array}{l} \hat{\theta} = \cos\theta \cos\phi \, \hat{i} + \cos\theta \sin\phi \, \hat{j} - \sin\theta \, \hat{k} \\ \hat{\phi} = -\sin\phi \, \hat{i} + \cos\phi \, \hat{j} \end{array} \right.$$

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$



$$\boxed{\vec{r} = r\,\hat{r} = r\left[\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}\right]}$$

•) $$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \boxed{\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}}$$

•) $$\hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\left[\cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}\right]$$

$$\left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{r^2\left(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right)} = r$$

$$\Rightarrow \boxed{\hat{\theta} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}}$$

•) $$\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$$

$$\frac{\partial r}{\partial \phi} = r\left[-\sin\theta\sin\phi\,\hat{\imath} + \sin\theta\cos\phi\,\hat{\jmath}\right]$$

$$\left|\frac{\partial r}{\partial \phi}\right| = \sqrt{r^2\left(\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi\right)} = r\sin\theta$$

$$\Rightarrow \boxed{\hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}}$$

$$\vec{L} = \overbrace{\frac{\hbar}{i}\left(-\sin\phi \frac{\partial}{\partial\theta} - \underbrace{\frac{1}{\sin\theta}\cos\theta}_{\cot\theta} \cos\phi \frac{\partial}{\partial\phi}\right)}^{L_x} \hat{i}$$

$$+ \overbrace{\frac{\hbar}{i}\left(\cos\phi \frac{\partial}{\partial\theta} - \underbrace{\frac{1}{\sin\theta}\cos\theta}_{\cot\theta} \sin\phi \frac{\partial}{\partial\phi}\right)}^{L_y} \hat{j}$$

$$+ \underbrace{\frac{\hbar}{i} \frac{\cancel{\sin\theta}}{\cancel{\sin\theta}} \frac{\partial}{\partial\phi}}_{L_z} \hat{k}$$

$$\boxed{L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}}$$

$$L_\pm = L_x \pm i L_y$$

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi}\right)$$

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)$$

$$L^2 = L_+L_- + L_z^2 - \hbar L_z$$

$$L^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \underbrace{\cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cancel{i\frac{\partial}{\partial\phi}} + \frac{\partial^2}{\partial\phi^2}}_{\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}} + \cancel{\frac{1}{i}\frac{\partial}{\partial\phi}}\right)$$

$$= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

$$\boxed{L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]}$$

we saw that the eigenvalues of L^2 were $\hbar^2 \ell(\ell+1)$:

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m$$

↳ but this is Eq (4.18) in the book → the angular equation leading the spherical harmonics Y_ℓ^m

↳ $\boxed{f_\ell^m = Y_\ell^m}$

Notice that, as we showed before, f_ℓ^m is an eigenfunction of L_z

$$L_z f_\ell^m = \hbar m f_\ell^m$$

↳ $\frac{\hbar}{i} \frac{\partial}{\partial\phi} f_\ell^m = \hbar m f_\ell^m$

Therefore: spherical harmonics are eigenfunctions of L^2 and L_z

H, L^2, L_z → commute, share the same eigenfunctions

$$H\psi = E\psi \qquad L^2\psi = \hbar^2 \ell(\ell+1)\psi \qquad L_z\psi = \hbar m\psi$$

Obs $m = -\ell, -\ell+1, \dots 0, \dots, \ell-1, \ell$

when studying L_+, L_-, we saw that maximum ℓ is obtained starting from $-\ell$ after N steps

$$-\ell + N = \ell \Rightarrow \boxed{\ell = N/2} \Rightarrow \ell \text{ is integer or half-integer}$$

this is different from what we obtained with the separation of variables (more comments later...)