Angular Momentum

$l, m \rightarrow$ related to angular momentum

$\vec{L} = \vec{r} \times \vec{p}$

$$\begin{tikzcd} x \arrow[r, bend left] & y \arrow[r, bend left] & z \arrow[ll, bend left] \end{tikzcd}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \hat{i}\,\underbrace{(y p_z - z p_y)}_{L_x} + \hat{j}\,\underbrace{(z p_x - x p_z)}_{L_y} + \hat{k}\,\underbrace{(x p_y - y p_x)}_{L_z}$$

$L_x, L_y, L_z \rightarrow$ incompatible, they cannot be determined at the same time
do not share the same eigenvectors

$$[L_x, L_y] = \left[(y p_z - z p_y), (z p_x - x p_z)\right] = y p_x \underbrace{[p_z, z]}_{-\hbar i} + p_y x \underbrace{[z, p_z]}_{\hbar i} =$$

$$= \hbar i \,(x p_y - y p_x) = \boxed{\hbar i \, L_z}$$

$\boxed{[L_y, L_z] = i\hbar L_x}$ $\qquad$ $\boxed{[L_z, L_x] = i\hbar L_y}$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \Bigg(\underbrace{\frac{\langle [L_x, L_y] \rangle}{2i}}_{\frac{\langle i\hbar L_z \rangle}{2i}}\Bigg)^2 \qquad \sigma_{L_x}^2 \sigma_{L_y}^2 \geq \frac{\hbar^2 \langle L_z \rangle^2}{4} \Rightarrow \boxed{\sigma_{L_x} \sigma_{L_y} \geq \frac{\hbar}{2} |\langle L_z \rangle|}$$

$$\boxed{\begin{tikzcd} L_x \arrow[r, bend left, "i\hbar L_z"] & L_y \arrow[r, bend left, "i\hbar L_x"] & L_z \arrow[ll, bend left, "i\hbar L_y"] \end{tikzcd}}$$

for the commutators

Square of the total angular momentum

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$\left[L^2, \begin{matrix} L_x \\ L_y \\ L_z \end{matrix}\right] = 0 \quad \longleftrightarrow \quad [L^2, \vec{L}] = 0$$

$$[L^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$$

$[AB, C] = A[B,C] + [A,C]B$

$$\downarrow\; = L_y \underbrace{[L_y, L_x]}_{-i\hbar L_z} + \underbrace{[L_y, L_x]}_{-i\hbar L_z} L_y + L_z \underbrace{[L_z, L_x]}_{i\hbar L_y} + \underbrace{[L_z, L_x]}_{i\hbar L_y} L_z = \boxed{0}$$

L^2 is compatible with each component L_x, L_y, L_z

$\rightarrow$) Let us pick L_z and find the eigenfunctions shared by L^2 and L_z

$$\boxed{L^2 f = \lambda f} \quad \text{and} \quad \boxed{L_z f = \mu f}$$

$\rightarrow$) to find λ, μ, we introduce the ladder operators

$$\boxed{L_\pm \equiv L_x \pm i L_y}$$

$$\boxed{[L^2, L_\pm] = 0}$$

$$[L_z, L_\pm] = \underbrace{[L_z, L_x]}_{i\hbar L_y} \pm i \underbrace{[L_z, L_y]}_{-i\hbar L_x} = \pm\hbar(L_x + i L_y) = \boxed{\pm \hbar L_\pm}$$

Therefore, if f is an eigenfunction of L^2, L_z

then $(L_\pm f)$ is also their eigenfunction, because

$$L^2 (L_\pm f) \overset{[L^2, L_\pm]=0}{=} L_\pm (L^2 f) = \boxed{\lambda (L_\pm f)}$$

$$L_z (L_\pm f) = L_\pm \underbrace{L_z f}_{\mu f} \pm \hbar L_\pm f = \boxed{(\mu \pm \hbar)(L_\pm f)}$$

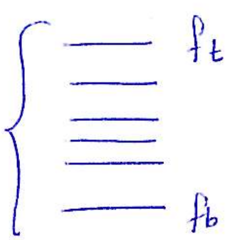
$\Downarrow$

L_+ : raising operator

L_- : lowering operator

→) For each value λ, there is a ladder of eigenstates of L_z with eigenvalues separated by $\hbar$

they are all degenerate in L^2, i.e, they all have the same eigenvalue λ



the lowest and highest eigenvalues of L_z are limited by λ

$$\begin{cases} L_z f_t = \mu_t f_t \\ L_+ f_t = 0 \end{cases}$$ because μ cannot be $> \lambda$

$$\begin{cases} L_z f_b = \mu_b f_b \\ L_- f_b = 0 \end{cases}$$

•) Let us write

$$L_z f_t = \hbar l f_t \qquad L_z f_b = \hbar \bar{l} f_b$$

and see how $\hbar l$ and $\hbar \bar{l}$ relate to λ

$$L_{\pm}L_{\mp} = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 \mp i\overbrace{(L_xL_y - L_yL_x)}^{i\hbar L_z} + L_y^2$$

$$= L_x^2 + L_y^2 + L_z^2 - L_z^2 \pm \hbar L_z$$

$$= L^2 - L_z^2 \pm \hbar L_z$$

$$\boxed{L^2 = L_{\pm}L_{\mp} + L_z^2 \mp \hbar L_z}$$

$$L^2 f_t = (\cancel{L_-L_+} + L_z^2 + \hbar L_z) f_t = (\hbar^2 l^2 + \hbar^2 l) f_t = \boxed{\hbar^2 l(l+1) f_t}$$

$$L^2 f_b = (\cancel{L_+L_-} + L_z^2 - \hbar L_z) f_b = (\hbar^2 \bar{l}^2 - \hbar^2 \bar{l}) f_b = \boxed{\hbar \bar{l}(\bar{l}-1) f_b}$$

since from f_b to f_t, they all have the same eigenvalue λ $\left\{ \begin{array}{l} L^2 f_t = \lambda f_t \\ \vdots \\ L^2 f_b = \lambda f_b \end{array} \right.$

$$\hookrightarrow\ l(l+1) = \bar{l}(\bar{l}-1) \Rightarrow \left\{ \begin{array}{l} \bar{l} = \cancel{l+1} \text{ which makes no sense} \\ \boxed{\bar{l} = -l} \end{array} \right.$$

$$L_z f_b = \hbar \bar{l} f_b = -\hbar l f_b$$

Raising

$$L_z(L_+ f_b) = L_+ L_z f_b + \hbar L_+ f_b = (-\hbar l + \hbar)(L_+ f_b)$$

$$\vdots$$

$$L_z f_t = \hbar l f_t$$

$$\hookrightarrow \boxed{L_z f = \hbar m f} \quad \text{where} \quad m = \underbrace{-l, -l+1, -l+2, \ldots, 0, 1, \ldots l-1, l}_{(2l+1)}$$

$$\boxed{L^2 f_l^m = \hbar^2 l(l+1) f_l^m} \qquad \boxed{L_z f_l^m = \hbar m f_l^m}$$

$$\boxed{f_l^m = Y_l^m}$$