Dirac Notation

$$\left.\begin{array}{l} |\alpha\rangle \\ \\ |S(t)\rangle \end{array}\right\} \text{basis is not specified}$$

VECTOR with respect to a BASIS

Once you write $|\alpha\rangle$ as $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$ $\rightarrow$ the basis $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$, $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}$, etc is specified

$\Psi(x,t) = \langle x | S(t) \rangle \rightarrow$ basis of position is specified

$\Phi(p,t) = \langle p | S(t) \rangle \rightarrow$ basis of momentum is specified

OPERATOR with respect to a BASIS

$$\langle e_m | \hat{Q} | e_n \rangle \equiv Q_{mn}$$

Example: in the basis $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ where $m_{11} = \langle e_1 | M | e_1 \rangle$

$m_{12} = \langle e_1 | M | e_2 \rangle$

$m_{21} = \langle e_2 | M | e_1 \rangle$

$m_{22} = \langle e_2 | M | e_2 \rangle$

Operator transforms one vector into another: $|\beta\rangle = \hat{Q}|\alpha\rangle$

$\rightarrow$ with respect to a basis, the equation becomes

$$\left.\begin{array}{l} |\alpha\rangle = \sum_n a_n |e_n\rangle, \quad a_n = \langle e_n | \alpha \rangle \\ |\beta\rangle = \sum_n b_n |e_n\rangle, \quad b_n = \langle e_n | \beta \rangle \end{array}\right\} \Rightarrow \sum_n b_n |e_n\rangle = \sum_n a_n Q |e_n\rangle \xRightarrow{\langle e_m|} \sum_n b_n \langle e_m | e_n \rangle = \sum_n a_n \langle e_m | Q | e_n \rangle$$

$$b_m = \sum_n Q_{mn} a_n$$

$$\begin{cases} |\alpha\rangle \to \text{ket} \to \text{vector space} \\ \quad\text{(column vector)} \\ \langle\beta| \to \text{bra} \to \text{dual space} \\ \quad\text{(row vector)} \end{cases}$$

↳ collection of all bras

- Suppose $|\alpha\rangle$ is a normalized vector

  $\hat{P} \equiv |\alpha\rangle\langle\alpha|$ is the <u>projector</u> operator

  it picks out the portion of any vector that lies along $|\alpha\rangle$

$$\hat{P}|\beta\rangle = |\alpha\rangle\langle\alpha|\beta\rangle = \langle\alpha|\beta\rangle\,|\alpha\rangle$$

- $\hat{P}$ is idempotent $\boxed{\hat{P}^2 = \hat{P}}$

(Prob 3.21)

$$\hat{P}^2|\beta\rangle = P\left(P|\beta\rangle\right) = P\left(\langle\alpha|\beta\rangle\,|\alpha\rangle\right) = \langle\alpha|\beta\rangle \underbrace{\langle\alpha|\alpha\rangle}_{1} |\alpha\rangle$$

$$= \langle\alpha|\beta\rangle\,|\alpha\rangle = \hat{P}|\beta\rangle$$

$$\hookrightarrow \hat{P}^2 = \hat{P}$$

- If $\{|e_n\rangle\}$ is a discrete <u>orthonormal</u> basis, that is, $\langle e_m|e_n\rangle = \delta_{mn}$

  then $\boxed{\sum_n |e_n\rangle\langle e_n| = \mathbb{1}}$ → identity operator

because if this operator acts on any vector $|\alpha\rangle$
$$\sum_n |e_n\rangle\underbrace{\langle e_n|\alpha\rangle}_{c_n} = \sum c_n|e_n\rangle = |\alpha\rangle$$
we recover $|\alpha\rangle$

Schwartz inequality

(Problem A 5)

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq | \langle \alpha | \beta \rangle |^2$$

Let

$$|\gamma\rangle = |\beta\rangle - \frac{\langle \alpha | \beta \rangle}{\langle \alpha | \alpha \rangle} |\alpha\rangle$$

since

$$\langle \gamma | \gamma \rangle \geq 0 \Rightarrow \left(\langle \beta | - \frac{\langle \alpha | \beta \rangle^*}{\langle \alpha | \alpha \rangle} \langle \alpha | \right) \left(|\beta\rangle - \frac{\langle \alpha | \beta \rangle |\alpha\rangle}{\langle \alpha | \alpha \rangle} \right)$$

$$= \langle \beta | \beta \rangle - \frac{\langle \alpha | \beta \rangle \langle \beta | \alpha \rangle}{\langle \alpha | \alpha \rangle} - \cancel{\frac{\langle \alpha | \beta \rangle^* \langle \alpha | \beta \rangle}{\langle \alpha | \alpha \rangle}} + \cancel{\frac{\langle \alpha | \beta \rangle \langle \alpha | \beta \rangle^* \cancel{\langle \alpha | \alpha \rangle}}{\langle \alpha | \alpha \rangle \cancel{\langle \alpha | \alpha \rangle}}} \geq 0$$

$$\hookrightarrow \langle \beta | \beta \rangle \geq \frac{\langle \alpha | \beta \rangle (\langle \alpha | \beta \rangle)^*}{\langle \alpha | \alpha \rangle}$$

$$\hookrightarrow \boxed{\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq | \langle \alpha | \beta \rangle |^2}$$

Uncertainty Principle

uncertainty
dispersion
standard deviation of observable A : σ_A (or uncertainty) / variance σ_A^2

$$\langle A^2\rangle - 2\langle A\rangle^2 + \langle A\rangle^2 = \langle A^2\rangle - \langle A\rangle^2$$

$$\sigma_A^2 = \langle (A-\langle A\rangle)^2\rangle = \langle \psi | (A-\langle A\rangle)^2 | \psi\rangle =$$

$$= \langle (A-\langle A\rangle)\psi | (A-\langle A\rangle)\psi\rangle = \langle f|f\rangle$$

$$\begin{cases} |f\rangle = (A-\langle A\rangle)|\psi\rangle \\ |g\rangle = (B-\langle B\rangle)|\psi\rangle \end{cases}$$

$$\sigma_B^2 = \langle (B-\langle B\rangle)^2\rangle = \langle g|g\rangle$$

(Schwartz inequality)

$$\rightarrow \sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geqslant |\langle f|g\rangle|^2$$

$$|z|^2 = [Re(z)]^2 + [Im(z)]^2 \geqslant [Im(z)]^2 = \left[\frac{z-z^*}{2i}\right]^2$$

$$\langle f|f\rangle\langle g|g\rangle \geqslant \left[\frac{\langle f|g\rangle - \langle f|g\rangle^*}{2i}\right]^2 \qquad \langle f|g\rangle^* = \langle g|f\rangle$$

$$\rightarrow \langle f|g\rangle = \langle (A-\langle A\rangle)\psi | (B-\langle B\rangle)\psi\rangle = \langle\psi| (A-\langle A\rangle)(B-\langle B\rangle)|\psi\rangle =$$

$$= \langle\psi| AB - A\langle B\rangle - \langle A\rangle B + \langle A\rangle\langle B\rangle |\psi\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle$$

$$\rightarrow \langle g|f\rangle = \langle BA\rangle - \langle A\rangle\langle B\rangle$$

$$\langle f|g\rangle - \langle f|g\rangle^* = \langle f|g\rangle - \langle g|f\rangle = \langle AB\rangle - \langle BA\rangle = \langle [A,B]\rangle$$

$$\boxed{\sigma_A^2\sigma_B^2 \geqslant \left(\frac{\langle [A,B]\rangle}{2i}\right)^2} \leftarrow$$

(generalized) uncertainty principle

$$\sigma_A^2 \sigma_B^2 = \left(\frac{\langle [A,B] \rangle}{2i}\right)^2$$

$[x, p] = i\hbar$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{i\hbar}{2i}\right)^2 = \frac{\hbar^2}{4}$$

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

Heisenberg uncertainty principle

uncertainty principle for every pair of $\begin{cases} \text{observables whose operators don't commute} \\ \text{incompatible observables} \end{cases}$

HW - Prob. 3.13

$\downarrow$

use test function

b) $[x^n, p] = i\hbar n x^{n-1}$

$$[x^n, p]\, g = x^n \frac{\hbar}{i}\cancel{\frac{\partial g}{\partial x}} - \frac{\hbar}{i} n x^{n-1} g - \cancel{\frac{\hbar}{i} x^n \frac{\partial g}{\partial x}} = -\frac{\hbar}{i} n x^{n-1} g$$

$$\Rightarrow \boxed{[x^n, p] = i\hbar n x^{n-1}}$$

Continuous Spectra

If the spectrum of a hermitian operator is continuous, the eigenfunctions are not normalizable

Ex. 3.2 eigenfunctions and eigenvalues of $\hat{p}$

$\underbrace{\text{eigenfunctions}}_{f_p(x)}$ and $\underbrace{\text{eigenvalues}}_{p}$

$$\frac{\hbar}{i} \frac{d}{dx} f_p(x) = p\, f_p(x)$$

$$\Rightarrow f_p(x) = A e^{ipx/\hbar} \longrightarrow \text{NOT square-integrable}$$

$$\boxed{\delta(p-p') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu(p-p')}\, du}$$

→) restrict to / assume / p - real

$u = x/\hbar$

$$\int_{-\infty}^{\infty} f_{p'}^{*}(x)\, f_p(x)\, dx = |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\, dx = |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')u}\, \hbar\, du$$

$$= |A|^2\, 2\pi\hbar\, \delta(p-p') \qquad \text{if} \quad \boxed{A = \frac{1}{\sqrt{2\pi\hbar}}}$$

$$\Rightarrow \int_{-\infty}^{\infty} f_{p'}^{*}(x)\, f_p(x)\, dx = \boxed{\langle f_{p'} | f_p \rangle = \delta(p-p')}$$

ORTHONORMAL

→) "orthonormal" non-normalizable

→) complete

Axiom

$$f(x) = \int_{-\infty}^{\infty} c(p)\, f_p(x)\, dp$$

to find $c(p)$

$$\boxed{c(p) = \langle f_p | f \rangle}$$

$$\langle f_{p'} | f \rangle = \int dx \int dp\, c(p) \frac{e^{-ip'x/\hbar}}{\sqrt{2\pi\hbar}} \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} = \int dp\, c(p)\, \delta(p-p') = \boxed{c(p')}$$

NOTE

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$ ← eigenfunction of the momentum operator

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \left[\cos\left(\frac{px}{\hbar}\right) + i\sin\left(\frac{px}{\hbar}\right)\right]$$

↳ to find the wavelength: $\frac{p\lambda}{\hbar} = 2\pi$

$$\boxed{\lambda = \frac{2\pi\hbar}{p}}$$ ⇒ The λ of the eigenfunctions of $\hat{p}$ is the de Broglie's λ.

Momentum vs. Position space

up to now, most of our studies were in position space

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, x\, \psi(x)\, dx$$

eigenfunctions of $\hat{p}$: $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

$$c(p) = \langle f_p | \psi \rangle$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx$$

$$\boxed{\phi(p,t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx}$$ ← $\phi(p,t)$ is the Fourier transform of $\Psi(x,t)$

momentum space wave function

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\, \phi(p,t)\, dp}$$ ← $\Psi(x,t)$ is the inverse Fourier transform of $\phi(p,t)$

position space wave function

Ex 3.4 A particle of mass m is bound in the delta function well

$V(x) = -\alpha\delta(x)$. What is the probability that a measurement of its momentum would yield a value greater than $p_0 = \frac{m\alpha}{\hbar}$?

$$\Psi(x,t) = \sqrt{\frac{m\alpha}{\hbar}}\, e^{-m\alpha|x|/\hbar^2}\, e^{-iEt/\hbar}$$

we need $\phi(p,t)$

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx = \frac{\sqrt{m\alpha}}{\hbar\sqrt{2\pi\hbar}}\, e^{-iEt/\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, e^{-m\alpha|x|/\hbar^2}\, dx$$

$$= \frac{\sqrt{m\alpha}}{\hbar\sqrt{2\pi\hbar}}\, e^{-iEt/\hbar} \left[\underbrace{\int_{-\infty}^{0} e^{x\left(-\frac{ip}{\hbar} + \frac{m\alpha}{\hbar^2}\right)} dx}_{\frac{e^{x(\dots)}\big|_{\infty}^{0}}{-\frac{ip}{\hbar} + \frac{m\alpha}{\hbar^2}}} + \underbrace{\int_{0}^{\infty} e^{x\left(-\frac{ip}{\hbar} - \frac{m\alpha}{\hbar^2}\right)} dx}_{\frac{e^{-x(\dots)}\big|_{0}^{\infty}}{-\frac{ip}{\hbar} - \frac{m\alpha}{\hbar^2}} = \frac{1}{\frac{ip}{\hbar} + \frac{m\alpha}{\hbar^2}}} \right]$$

$$= \frac{\sqrt{m\alpha}}{\hbar\sqrt{2\pi\hbar}}\, e^{-iEt/\hbar} \left[\frac{+\cancel{ip/\hbar} + m\alpha/\hbar^2 - \cancel{ip/\hbar} + m\alpha/\hbar^2}{\left(p^2/\hbar^2 + m^2\alpha^2/\hbar^4\right)} \right]$$

$$= \frac{\sqrt{m\alpha}}{\hbar\sqrt{2\pi\hbar}}\, e^{-iEt/\hbar}\, \frac{2m\alpha}{\left(p^2 + m^2\alpha^2/\hbar^2\right)} = \sqrt{\frac{2}{\pi}}\, e^{-iEt/\hbar}\, \frac{\sqrt{(m\alpha/\hbar)^3}}{p^2 + (m\alpha/\hbar)^2} = \boxed{\sqrt{\frac{2}{\pi}}\, \frac{p_0^{3/2}\, e^{-iEt/\hbar}}{p^2 + p_0^2}}$$

probability → measurement greater than p_0

$$\int_{p_0}^{\infty} |\phi(p,t)|^2\, dp = \frac{2}{\pi}\, p_0^3 \int_{p_0}^{\infty} \frac{1}{(p^2 + p_0^2)}\, dp = \boxed{0.0908}$$

Prob. 3.10

Is the ground state of the infinite square well an eigenfunction of momentum?

$$\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

$$\hat{p}\,\psi_1 = \frac{\hbar}{i}\sqrt{\frac{2}{a}}\,\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right) = \frac{\hbar}{i}\frac{\pi}{a}\,\mathrm{cotg}\left(\frac{\pi x}{a}\right)\boxed{\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)} = \underbrace{\frac{\hbar}{i}\frac{\pi}{a}\,\mathrm{cotg}\left(\frac{\pi x}{a}\right)}_{\text{NOT a const}}\psi_1$$

↳ ψ_1 is NOT an eigenfunction of $\hat{p}$

Prob. 3.11

Find $\phi(p,t)$ for a particle in the ground state of the harmonic oscillator

$$\psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\, e^{-i\omega t/2}$$

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\psi_0(x,t)\,dx \underset{\text{(complete square ...)}}{=} \frac{1}{(\pi m\omega\hbar)^{1/4}}\, e^{-p^2/2m\omega\hbar}\, e^{-i\omega t/2}$$

What is the prob. that a measurement of p on a particle in this state would yield a value outside the classical range (for the same energy)?

maximum p classically } all E is kinetic } $\quad E = \frac{\hbar\omega}{2} = \frac{p^2}{2m} \Rightarrow p = \pm\sqrt{m\hbar\omega}$

$$\int_{-\infty}^{-\sqrt{m\hbar\omega}} |\phi(p,t)|^2\,dp + \int_{\sqrt{m\hbar\omega}}^{\infty} |\phi(p,t)|^2\,dp = \boxed{0.16}$$

(use mathematica)

Prob. 3.12

$$\langle x\rangle = \int \phi^*(p,t)\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\phi(p,t)\,dp$$

$$\langle x\rangle = \int \Psi^*\, x\, \Psi\, dx =$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{ipx/\hbar}\,\phi(p,t)\,dp$$

$$= \frac{1}{2\pi\hbar}\int dp'\int dp\int dx\; e^{-ip'x/\hbar}\,\phi^*(p',t)\;\underbrace{x\, e^{ipx/\hbar}\,\phi(p,t)}_{\frac{\hbar}{i}\frac{\partial}{\partial p}e^{ipx/\hbar}}$$

$$= \frac{1}{2\pi\hbar}\int dp'\int dp\int dx\; e^{-ip'x/\hbar}\,\phi^*(p',t)\;\frac{\hbar}{i}\left(\frac{\partial}{\partial p}e^{ipx/\hbar}\right)\phi(p,t)$$

$$= \frac{1}{2\pi\hbar}\int dp'\int dx\; e^{-ip'x/\hbar}\,\phi^*(p',t)\;\frac{\hbar}{i}\left[\cancel{e^{ipx/\hbar}\,\phi(p,t)\Big|_{-\infty}^{\infty}} - \int e^{ipx/\hbar}\frac{\partial}{\partial p}\phi(p,t)\,dp\right]$$

$$= \underbrace{\frac{1}{2\pi\hbar}\int dp'\int dp\;\phi^*(p',t)\int dx\; e^{i(p-p')x/\hbar}}_{\delta(p-p')}\left[\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\phi(p,t)\right]$$

$$= \int dp\;\phi^*(p,t)\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\phi(p,t)$$

$$\left\{\; Q(\hat{x},\hat{p}) \longrightarrow \begin{cases} \nearrow\; Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right) \\ \searrow\; Q\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p\right)\end{cases}\right.$$