EIGENVALUES and EIGENVECTORS

3D – special vectors where rotation $\Rightarrow$ $T|\alpha\rangle = |\alpha\rangle$
(vector along the rotation axis)

$\Rightarrow$ $T|\alpha\rangle = -|\alpha\rangle$
(vector along $\perp$ axis and $\theta = 180°$)

In complex vector space – also find such special vector

$$T|\alpha\rangle = \underbrace{\lambda}_{\text{eigenvalues (complex numbers)}} \underbrace{|\alpha\rangle}_{\text{eigenvectors}}$$

- spectrum of a matrix/operator – collection of all its eigenvalues
- if 2 or more eigenvectors share the same eigenvalue $\Rightarrow$ the spectrum is DEGENERATE

$\rightarrow$) Time-indep. Schröd. eq.

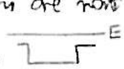
$$\boxed{H\psi = E\psi}$$

E: eigenval.
ψ: eigenvect. / eigenfunctions

$\rightarrow$) spectrum can be DISCRETE or CONTINUOUS

(1) discrete : eigenvalues are separated from one another $\Rightarrow$ eigenfunctions lie in Hilbert space and are physically realizable states
$\downarrow$
Example: H.O. (discrete and infinite dimensional spectrum), infinite square well, delta potential $E<0$
spin chain with L sites (discrete and finite dimensional spectrum)

(2) continuous : eigenvalues fill out an entire range $\Rightarrow$ eigenfunctions are not normalizable [but linear combinations may be normalizable]
Example: free particle, delta potential well with $E>0$,



(!) important

DISCRETE

Normalizable eigenfunctions of a hermitian operator have the properties

$\boxed{H\psi = E\psi}$

o) Eigenvalues are REAL (as E in $H\psi = E\psi$)

hermitian operators $\Leftrightarrow$ $\langle T \rangle = \langle T \rangle^*$

$$\langle \psi | T\psi \rangle = \langle T\psi | \psi \rangle$$
$$t\langle \psi|\psi\rangle = t \qquad t^*\langle\psi|\psi\rangle = t^*$$

$\langle T \rangle = \langle T \rangle^*$

$$\langle T \rangle = \langle \psi | T | \psi \rangle \underset{T|\psi\rangle = t|\psi\rangle}{=} t\langle \psi | \psi \rangle \underset{\text{norm}}{=} t$$

$$\langle T \rangle^* = \langle \psi | T | \psi \rangle^* = t^*$$

$\langle T \rangle = \langle T \rangle^* \Rightarrow \boxed{t = t^*}$

o) Eigenfunctions belonging to distinct eigenvalues are ORTHOGONAL

as $\langle \psi_m | \psi_n \rangle = 0$ in infinite square well

$$T\psi_1 = t_1\psi_1 \qquad T\psi_2 = t_2\psi_2$$

T is hermitian

$$\langle \psi_1 | T\psi_2 \rangle = \langle T\psi_1 | \psi_2 \rangle$$
$$t_2\langle \psi_1|\psi_2\rangle \qquad t_1^*\langle \psi_1|\psi_2\rangle$$
$$= t_1\langle \psi_1|\psi_2\rangle \quad \text{from above}$$

since $t_1 \neq t_2$ by assumption

$\Rightarrow \langle \psi_1 | \psi_2 \rangle = 0$

o) finite-dim $\rightarrow$ eigenvectors of a hermitian matrix SPAN the space

$\Rightarrow$ any vector can be expressed as a linear comb of them

take it as an AXIOM for infinite-dim

(= proposition assumed without proof)

$$\Psi(x,t) = \sum c_n \psi_n(x) e^{-iE_nt/\hbar}$$ for $\hat{H}$

DISCRETE SPECTRUM and FINITE DIMENSIONAL

How to find eigenvectors

$$T|\alpha\rangle = \lambda|\alpha\rangle$$

$$(T - \mathbb{1}\lambda)|\alpha\rangle = 0 \leftarrow \text{zero matrix}$$

if it had inverse $\underbrace{(T-\lambda\mathbb{1})^{-1}(T-\lambda\mathbb{1})}_{\mathbb{1}}|\alpha\rangle = 0 \Rightarrow |\alpha\rangle = 0$

but assume $|\alpha\rangle \neq 0$

$$\Rightarrow \det(T-\lambda\mathbb{1}) = 0 \Leftrightarrow (T-\lambda\mathbb{1}) \text{ has no inverse}$$

$$\Downarrow$$

$$\boxed{\text{characteristic equation}}$$

Example:

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$$

$$\det(M-\lambda\mathbb{1}) = \begin{vmatrix} 2-\lambda & 0 & -2 \\ -2i & i-\lambda & 2i \\ 1 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(i-\lambda)(-1-\lambda) + 2(i-\lambda) = 0$$

$$\Rightarrow (2-\lambda)(-i-i\lambda+\lambda+\lambda^2) + 2i - 2\lambda = \cancel{-2i} - 2i\lambda + \cancel{2\lambda} + 2\lambda^2 + \lambda i + i\lambda^2 - \lambda^2 - \lambda^3 + \cancel{2i} - \cancel{2\lambda}$$

$$= -\lambda^3 + (1+i)\lambda^2 - i\lambda = 0$$

$$\left\{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = i \end{matrix}\right\} \quad -\lambda(\lambda^2 - (1+i)\lambda + i) = 0 \rightarrow \lambda = 0$$

$$\lambda = \frac{(1+i) \pm \sqrt{(\cancel{1} + 2i - \cancel{1}) - 4i}}{2}$$

$$-\lambda^3 + (1+i)\lambda^2 - i\lambda = 0$$

$$\Rightarrow -\lambda\left(\lambda^2 - (1+i)\lambda + i\right) = 0$$

$$\lambda^2 - (1+i)\lambda + i = 0 \Rightarrow \lambda = \frac{(1+i) \pm \sqrt{(\cancel{1} + 2i - \cancel{1}) - 4i}}{2}$$

$$\lambda = \frac{(1+i) \pm \sqrt{-2i}}{2} = \frac{(1+i) \pm i\sqrt{2}\sqrt{i}}{2} = \frac{(1+i) \pm i\cancel{\sqrt{2}}(1+i)/\cancel{\sqrt{2}}}{2}$$

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$\lambda = \frac{1+i \pm (i-1)}{2} \quad \begin{matrix} \nearrow i \\ \searrow 1 \end{matrix}$$

$$\boxed{\lambda = \begin{cases} 0 \\ 1 \\ i \end{cases}}$$

For $\lambda_1 = 0$

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left.\begin{cases} 2x - 2z = 0 & \Rightarrow x = z \\ -2ix + iy + 2iz = 0 & \\ x - z = 0 & \Rightarrow x = z \end{cases}\right\} \Rightarrow y = 0 \quad \Rightarrow |\alpha\rangle = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix}$$

from $\langle\alpha|\alpha\rangle = 1 \Rightarrow (x^* \;\; 0 \;\; x^*)\begin{pmatrix} x \\ 0 \\ x \end{pmatrix} =$

$\Rightarrow |x|^2 = 1/2$

$\boxed{\lambda_1 = 0}$ and $\boxed{|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$ //

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} 2x - 2z = x & \longrightarrow \quad x = 2z \Rightarrow \overline{z = x/2} \\ -2ix + iy + 2iz = y & \longrightarrow \quad -2ix + ix = y - iy \\ x - z = z & \end{cases}$$

$$-ix = y(1-i)$$

$$\underline{\overline{y = \frac{i}{(i-1)} x}}$$

$$\lambda_2 = 1 \longrightarrow \psi_2 = \begin{pmatrix} x \\ \frac{i}{(i-1)} x \\ x/2 \end{pmatrix}$$

$$-(i+1)(i-1) = -(-1-1)$$

$$\psi_2^* \psi_2 = \begin{pmatrix} x^* & \frac{-i}{(-i-1)} x^* & \frac{x^*}{2} \end{pmatrix} \begin{pmatrix} x \\ \frac{i}{(i-1)} x \\ \frac{x}{2} \end{pmatrix} = 1$$

$$|x|^2 \left(1 + \frac{1}{2} + \frac{1}{4} \right) = |x|^2 \, \frac{7}{4} = 1 \longrightarrow x = \frac{2}{\sqrt{7}}$$

$$\boxed{\overline{\underline{\lambda_2 = 1}} \Rightarrow \psi_2 = \frac{2}{\sqrt{7}} \begin{pmatrix} 1 \\ \frac{i}{(i-1)} \\ \frac{1}{2} \end{pmatrix} = \frac{2}{\sqrt{7}} \begin{pmatrix} 1 \\ \frac{1-i}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix}}$$

$$\frac{i}{i-1} = \frac{i(i+1)}{(-1-1)} = \frac{-1+i}{-2} = \frac{1-i}{2}$$

$\lambda_3 = i$ $\underline{\text{etc}}$

Changing Basis

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}$$

same matrices but written in different basis } similar matrices

M uses $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

M' uses ψ_1, ψ_2, ψ_3

while the elements may look very $\neq$ in the new basis

det and trace don't change

$\downarrow$

(sum of diagonal elements)

$\rightarrow$) check trace of M, M'

$\rightarrow$) check det

how numbers change as we change basis
↓
in vector
in matrix

Vector

$$\boxed{|\alpha^f\rangle = S|\alpha^e\rangle}$$

matrix

$$|\alpha'^e\rangle = T^e|\alpha^e\rangle \quad \text{and} \quad |\alpha'^f\rangle = T^f|\alpha^f\rangle$$

$$|\alpha'^f\rangle = S|\alpha'^e\rangle = S\,(T^e\underbrace{|\alpha^e\rangle}_{S^{-1}|\alpha^f\rangle}) = ST^eS^{-1}|\alpha^f\rangle$$

$$\shortparallel$$

$$|T^f|\alpha^f\rangle$$

$$\boxed{T^f = ST^eS^{-1}}$$

S – similarity matrix that effects the diagonalization is constructed by using the normalized eigenvectors as the columns of S^{-1}

$$S^{-1} = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

A NORMAL matrix commutes with its hermitian conjugate

$$[N^\dagger, N] = 0$$

Every normal matrix is diagonalizable

Example: hermitian matrix, unitary matrix

→) ~~any~~ two matrices that $\boxed{\text{commute}}$ ~~are~~ can be simultaneously diagonalized by the same (similarity) matrix S

→) or if they are simultaneously diagonalized ⇒ they commute

$$S A S^{-1} = D \Rightarrow \underline{A = S^{-1} D S} \qquad S B S^{-1} = E \Rightarrow \underline{B = S^{-1} E S}$$

$$[A,B] = AB - BA = S^{-1}DSS^{-1}ES - S^{-1}ESS^{-1}DS = S^{-1}DES - S^{-1}EDS = S^{-1}\underbrace{[D,E]}_{\text{diagonal always commute}}S = 0$$

→) operators that ◦) commute are COMPATIBLE observables
and share eigen[vectors / functions]

◦) don't commute are INCOMPATIBLE observables
don't share ψ's
↳ like x and p

→) The hermitian conjugate (or adjoint) of an operator $\hat{Q}$ is the operator $\hat{Q}$ such that

operators
$$\begin{cases} \langle \alpha | \hat{Q} \beta \rangle = \langle \hat{Q}^{+} \alpha | \beta \rangle \\ \langle \alpha | \hat{Q}^{+} \beta \rangle = \langle Q \alpha | \beta \rangle \end{cases}$$

but
The operator is hermitian if $\overbrace{Q^{+} = Q}^{\text{finite dim}}$
that is
$$\boxed{\langle \alpha | Q \beta \rangle = \langle Q \alpha | \beta \rangle}$$

orthonormal basis $\langle \alpha |$ – row $| \beta \rangle$ – column

$$\underline{\langle \alpha | \hat{Q} | \beta \rangle} = a^{+} Q b = (Q^{+} a)^{+} b = \underline{\langle \hat{Q}^{+} \alpha | \beta \rangle}$$

Generalized Statistical Interpretation

We saw that

$$\underset{\text{hermitian}}{H}\psi_n = \underset{\text{eigenvalues}}{E_n}\underset{\text{eigenfunctions}}{\psi_n}$$

$$\langle \psi_m | \psi_n \rangle = \delta_{nm} \implies \Psi(x,t) = \sum_1 c_n \psi_n(x)\, e^{-iE_n t/\hbar}$$

$$\boxed{\langle H \rangle = \sum_1 |c_n|^2 E_n} \quad \begin{cases} |c_1|^2 \text{ prob. for } E_1 \\ |c_2|^2 \text{ prob. for } E_2 \\ \vdots \end{cases}$$

$$\boxed{\sum_1 |c_n|^2 = 1}$$

Let's extend this to any observable

If we measure an Observable $Q(x,p)$ on a particle in state $\Psi(x,t)$, we get one eigenvalue of the hermitian operator $\hat{Q}(x, \frac{\hbar}{i}\frac{d}{dx})$

a) spectrum of $\hat{Q}$ is discrete

prob. of getting eigenvalue q_n associated with eigenfunction $f_n(x)$ is

$$\boxed{|c_n|^2} \quad \text{where} \quad c_n = \langle f_n | \Psi \rangle$$

c) spectrum is continuous

prob. is in the range dz for getting $q(z)$ associated with $f_z(x)$

$$\boxed{|c(z)|^2\, dz} \quad \text{where} \quad c(z) = \langle f_z | \Psi \rangle$$

$$\Psi(x,t) = \sum_n c_n(t)\, f_n(x)$$

$\rightarrow$) $\sum_n |c_n|^2 = 1$

$$1 = \langle \Psi | \Psi \rangle = \left\langle \sum c_{n'}^* f_{n'}^* \middle| \sum c_n f_n \right\rangle$$

$$= \sum_{n'} \sum_n c_{n'}^* c_n \underbrace{\langle f_{n'} | f_n \rangle}_{\delta_{n'n}}$$

$$= \boxed{\sum_n |c_n|^2} = 1$$

$\rightarrow$) $\langle Q \rangle = \sum_n q_n |c_n|^2$

$$\langle Q \rangle = \langle \Psi | Q | \Psi \rangle = \left\langle \sum_{n'} c_{n'}^* f_{n'} \middle| \underbrace{Q \sum c_n f_n}_{q_n f_n} \right\rangle$$

$$= \sum_{n'} \sum_n c_{n'}^* c_n q_n \underbrace{\langle f_{n'} | f_n \rangle}_{\delta_{n'n}}$$

$$= \boxed{\sum_n |c_n|^2 q_n}$$

NOTE: Upon measurement, the wave function COLLAPSES to the corresponding eigenstate

Example: operator $\hat{A}$ has two normalized eigenstates ϕ_1 and ϕ_2 with eigenvalues a_1 and a_2.
If observable A is measured and we get a_1, the state of the system (which could have been a superposition of ϕ_1 and ϕ_2) immediately collapses and becomes $\boxed{\phi_1}$ //

Example 3.8

$$H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$$ basis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$E_+ = h + g \qquad |s_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_- = h - g \qquad |s_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If the system starts out in state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is its state at time t?

$$|s(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|s_+\rangle + |s_-\rangle \right)$$

$$|s(t)\rangle = \frac{1}{\sqrt{2}} \left(|s_+\rangle e^{-iE_+t/\hbar} + |s_-\rangle e^{-iE_-t/\hbar} \right)$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-iE_+t/\hbar} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-iE_-t/\hbar} = e^{-iht/\hbar} \begin{pmatrix} \frac{e^{-igt/\hbar} + e^{+igt/\hbar}}{2} \\ \frac{e^{-igt/\hbar} - e^{+igt/\hbar}}{2} \end{pmatrix}$$

$$= e^{-iht/\hbar} \begin{pmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{pmatrix}$$

at $t=0 \implies |s(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (phase)

at $\frac{gt}{\hbar} = \frac{\pi}{2} \implies |s\left(\frac{\pi\hbar}{2g}\right)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ phase

at $\frac{gt}{\hbar} = \pi \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ phase