Delta Function Poincare

Up to now: two kinds of solutions to time-independent Schröd. eq.

\[ \text{(i) Infini square well} \quad \text{normalizable, labeled by discrete index n (physically realizable)} \]

\[ \text{(ii) Free particle} \quad \text{non-normalizable, labeled by continuous variable k (not physically realizable)} \]

Both cases: general solution to time-dependent Schröd. eq. \( \rightarrow \) linear combination of stationary states

\[ \text{I) sum over n} \]
\[ \text{II) integral over k} \]

Compare with classical mechanics

If \( V > E \) on either side
Particle is "shack"ed in between the turning points
\( \Rightarrow \) bound state

If \( E > V \) on either side
Particle comes from infinity, slows down, and returns to infinity
\( \Rightarrow \) scattering state
1) Equivalently in quantum mechanics

\[ \text{harm. oscil. and inf. square well} \rightarrow \begin{cases} \text{bound state} & (E < V) \\ \text{free particle} & (E > V) \end{cases} \]

\[ \text{What is surprising here:} \]

1) particle can leak through \underline{finite} potential barrier
   (like in the case of the harm. oscil.)
   \[ \rightarrow \text{tunneling} (E < V) \]

2) particle can scatter even when \( E > V(200) \)

\[ \text{Examples} \]

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<td>particle coming from ( -\infty )</td>
<td>( V(x) )</td>
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<td>( \text{particle is reflected} )</td>
<td>( \text{there is reflection and transmission (tunneling)} )</td>
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In QM, particles can "leak" through finite potential = tunneling

For potentials that go to zero at infinity

\[ v(\infty) \]

\[ \begin{align*} 
E < 0 & \Rightarrow \text{bound state} \\
E > 0 & \Rightarrow \text{scattering state} 
\end{align*} \]

\[ v(\infty) \]

Dirac delta function

\[ \delta(x) = \begin{cases} 
0 & \text{if } x \neq 0 \\
\infty & \text{if } x = 0 
\end{cases} \]

\[ \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \]

\[ \int_{-\infty}^{\infty} f(x) \delta(x-a) \, dx = f(a) \]

\[ \int_{a-\delta}^{a+\delta} \delta(x-a) \, dx = 1 \]

Consider

\[ V(x) = -\delta(x) \]

\[ \frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \delta(x) \psi = E \psi \]

It yields

\[ \begin{cases} 
\text{bound states (E < 0)} \\
\text{scattering states (E > 0)} 
\end{cases} \]
Bound state \( E \leq 0 \)

\[ x < 0 \quad \Rightarrow \quad V(x) = 0 \]

\[ \frac{d^2 \psi}{dx^2} = -\frac{2mE}{\hbar^2} \psi = \mp K^2 \psi \]

\[ K = \sqrt{-\frac{2mE}{\hbar}} \quad \Rightarrow \quad E \leq 0 \Rightarrow K > 0 \]

\[ \psi(x) = A e^{-Kx} + B e^{Kx} \]

\[ \downarrow \]

blows up at \( x \to -\infty \) \( \Rightarrow A = 0 \)

\[ \psi(x) = B e^{Kx} \quad (x < 0) \]

\[ \longrightarrow \]

\[ x > 0 \quad \Rightarrow \quad V(x) = 0 \]

\[ \psi(x) = F e^{-Kx} + G e^{Kx} \]

\[ \downarrow \]

blows up at \( x \to +\infty \)

\[ \psi(x) = F e^{-Kx} \quad (x > 0) \]

How to connect the two solutions using boundary conditions at \( x = 0 \)...

1) \( \psi \) is always continuous

\[
\psi(x) = \begin{cases} 
Be^{Kx} & (x \leq 0) \\
Be^{-Kx} & (x > 0)
\end{cases}
\]

\[K = ?\]

Integrate Schröd eq. around zero (from \(-\epsilon\) to \(\epsilon\) with \(\epsilon \to 0\))

\[
\frac{-\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} \frac{d^2 \psi}{dx^2} \, dx + \int_{-\epsilon}^{\epsilon} V(x) \psi(x) \, dx = E \int_{-\epsilon}^{\epsilon} \psi(x) \, dx
\]

\[
\lim_{\epsilon \to 0} \frac{d\psi}{dx} \bigg|_{x=\epsilon} = \frac{2m}{\hbar^2} \int_{-\epsilon}^{\epsilon} (-\Delta) \delta(x) \psi(x) \, dx
\]

\[
\lim_{\epsilon \to 0} \frac{d\psi}{dx} \bigg|_{x=-\epsilon} = \frac{2m}{\hbar^2} \int_{-\epsilon}^{\epsilon} (-\Delta) \delta(x) \psi(x) \, dx
\]

\[
\Delta \frac{d\psi}{dx} = -\frac{2m \lambda}{\hbar^2} \psi(0)
\]

\[
(x > 0) \quad \frac{d\psi}{dx} = -BK e^{-Kx} \quad \Rightarrow \quad \frac{d\psi}{dx} \bigg|_{x=0} = -BK
\]

\[
(x < 0) \quad \frac{d\psi}{dx} = BK e^{Kx} \quad \Rightarrow \quad \frac{d\psi}{dx} \bigg|_{x=0} = +BK
\]

\[
-2BK = -\frac{2m \lambda}{\hbar^2} B \quad \Rightarrow \quad K = \frac{m \lambda}{\hbar^2}
\]
\[ E = -\frac{h^2 k^2}{2m} = \frac{-m \Delta^2}{2 \hbar^2} \]  

allowed energy

Nomdeny \( \Psi \)

\[
\int_{0}^{\infty} |\Psi(x)|^2 \, dx = 21B^2 \int_{0}^{\infty} e^{-2kx} \, dx = \\
= 21B^2 \frac{e^{-2kx}}{-2k} \bigg|_{0}^{\infty} = \frac{21B^2}{2k} = 1
\]

\[ B = \sqrt{K} = \sqrt{\frac{m \Delta}{h}} \]

\[ \Psi(x) = \frac{\sqrt{m \Delta}}{h} e^{-\frac{m|\Delta| x}{h^2}} ; \quad E = -\frac{m \Delta^2}{2 \hbar^2} \]
Scattering States \( E > 0 \)

\[ x < 0 \ (E < 0) \] \[
-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E \psi
\]

\[
k^2 = \frac{2mE}{\hbar^2} \implies E = \frac{k^2 \hbar^2}{2m}
\]

\[ \psi(x) = A e^{ikx} + B e^{-ikx} \]

\[ \phi(x) = F e^{ikx} + G e^{-ikx} \]

**Continuity of \( \psi(x) \) at \( x = 0 \)**

\[ A + B = F + G \]

**Can't use continuity of \( \frac{d\psi}{dx} \) at \( x = 0 \), because at this point \( V \) is infinite**

So we integrate Schröd. eq. from \(-\epsilon\) to \(\epsilon\) with \(\epsilon \to 0\)

\[-\frac{\hbar^2}{2m} \left( \frac{d\psi}{dx} \bigg|_{x^+} - \frac{d\psi}{dx} \bigg|_{x^-} \right) - \lambda \psi(0) = E \left( \psi(\epsilon) - \psi(-\epsilon) \right) \]

\[ iK (F-G) - iK (A-B) = -\frac{2m}{\hbar^2} (A+B) \]

\[ (F-G) = \frac{-2m}{\hbar^2} iK (A+B) + (A-B) \implies F-G = A(1+2i\beta) - B(1-2i\beta) \]

\( \text{2 equations and 5 unknowns} \)
physically

\[
\begin{align*}
A & = e^{-ikx} \\
B & = e^{ikx} \\
G & = e^{-ikx} \\
F & = e^{ikx}
\end{align*}
\]

\[
\begin{align*}
A : & \text{ amplitude of a wave coming from the left } (-\infty) \\
B : & \text{ " returning to } -\infty \\
G : & \text{ " coming from the right } (+\infty) \\
F : & \text{ " returning to } +\infty
\end{align*}
\]

Observing particles are fired from one direction

Assume from the left \rightarrow scattering from the left \rightarrow \begin{cases} G = 0 \end{cases}

A is the amplitude of the incident wave

B is the amplitude of the reflected wave

F is the amplitude of the transmitted wave

\[
\begin{cases}
A + B = F \\
F = A(1+2i\beta) - B(1-2i\beta)
\end{cases}
\]

\[
\Rightarrow \begin{cases} B = \frac{i\beta}{1-i\beta} A \\
F = \frac{1}{1-i\beta} A \end{cases}
\]

\[\text{Reflection coefficient} \quad R = \frac{|B|^2}{|A|^2}\]

\[\text{Transmission coefficient} \quad T = \frac{|F|^2}{|A|^2}\]

\[R + T = 1\]
\[
R = \frac{1}{1 + \frac{2\hbar^2 E}{m\alpha^2}} \\
T = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}}
\]

higher energy \(\rightarrow\) greater prob. of transmission (longer \(T\))
smaller energy \(\rightarrow\) greater \(R\)

\[
\begin{align*}
A + B &= F \\
F &= A(1 + 2i\beta) - B(1 - 2i\beta) \\
2B(1 \circ i\beta) &= 2i\beta A \\
B &= \frac{i\beta}{1-i\beta} \\
F &= \frac{1}{1-i\beta} A
\end{align*}
\]

\[
R = \frac{\beta^2}{1+\beta^2} \\
\beta = \frac{m\alpha}{\hbar \sqrt{2mE}} \\
E = \frac{\hbar^2 \kappa^2}{2m} \\
T = \frac{1}{1+\beta^2}
\]

\[
R = \frac{\frac{\beta^2}{\hbar^2}}{\frac{2E}{\hbar^2 E + m\alpha^2}} \\
\rightarrow R = \frac{1}{1 + \frac{2\hbar^2 E}{m\alpha^2}} \\
T = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^4 E}}
\]