Delta Function Potential

Up to now: two kinds of solutions to time-indep. Schrö'd. eq.

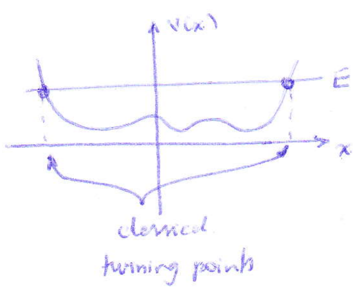
Ⅰ Infinite square well, Harmonic Oscillator } normalizable, labeled ~~with~~ by discrete index n (physically realizable)

Ⅱ Free particle } non-normalizable, labeled by continuous variable k (not physically realizable)

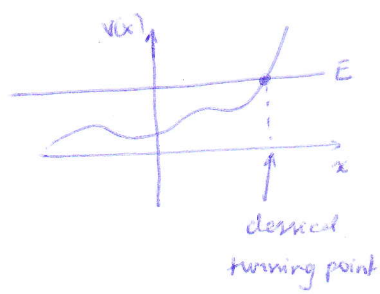
Both cases: general solution to time-dep Schröd. eq. → linear combination of stationary states
↓
Ⅰ sum over n
Ⅱ integral over k

Compare with classical mechanics

if V > E on either side
particle is "stuck" in
between the turning points
⇒ bound state



if E > V on either side
particle comes from infinity,
slows down, and returns to
infinity
⇒ scattering state



→) Equivalently in quantum mechanics

harm. oscill. and infinite square well } bound state (E<V)

free particle } scattering state (E>V)

What is surprising here:

·) particle can leak through FINITE potential barrier
(like in the case of the harm. oscill.)

⇓

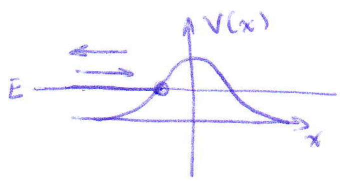
tunneling $(E < V_{(\pm\infty)})$

·) particle can scatter even when $E > V(\pm\infty)$

Examples

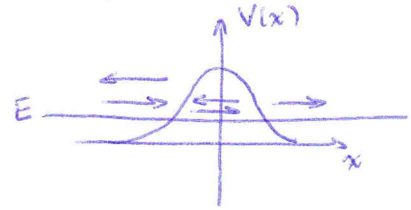
classically

Particle coming from $-\infty$

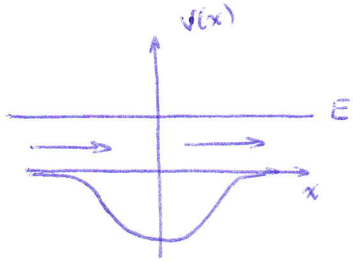


particle is reflected
transmission = 0

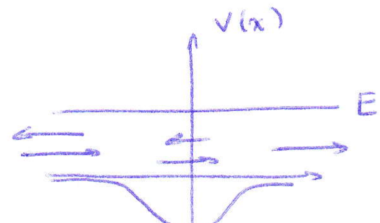
quantum mechanically



there is reflection and transmission (tunneling)



particle is transmitted
reflection = 0



there is reflection and transmission

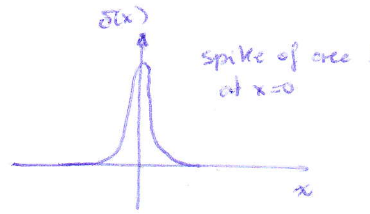
In Q.M. particle can "leak" through finite potential $\rightsquigarrow$ tunneling

For potentials that go to zero at infinity

$$\begin{cases} \underset{V(\pm\infty)}{E<0} \Rightarrow \text{bound state} \\ \underset{V(\pm\infty)}{E>0} \Rightarrow \text{scattering state} \end{cases}$$

Dirac delta function

$$\delta(x) \equiv \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \xrightarrow{\text{with}} \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$



$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

picks out the value of $f(x)$ at point a

$$\xrightarrow{\text{also}} \int_{a-\varepsilon}^{a+\varepsilon} f(x)\, \delta(x-a)\, dx = f(a)$$

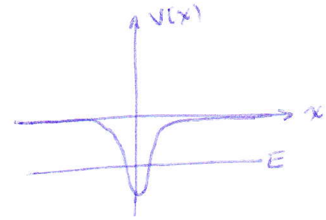
domain of integration includes a

Consider

$$\boxed{V(x) = -\alpha\, \delta(x)} \qquad \alpha > 0$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\psi$$

it yields $\begin{cases} \text{bound states } (E<0) \\ \text{and} \\ \text{scattering states } (E>0) \end{cases}$



Bound states $\boxed{E<0}$

→) $x<0 \Rightarrow V(x)=0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = (+)K^2\psi$$

$$K = \frac{\sqrt{-2mE}}{\hbar} \qquad E<0 \Rightarrow K\ (+)$$

$$\psi(x) = \underset{\downarrow}{Ae^{-Kx}} + Be^{Kx}$$

blows up at $x \to -\infty \Rightarrow A=0$

$$\boxed{\psi(x) = Be^{Kx}} \qquad (x<0)$$

→) $x>0 \Rightarrow V(x)=0$

$$\psi(x) = Fe^{-Kx} + \underset{\downarrow}{Ge^{Kx}}$$

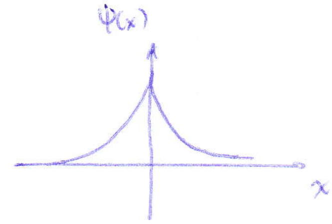
blows up at $x \to +\infty$

$$\boxed{\psi(x) = Fe^{-Kx}} \qquad (x>0)$$

How to connect the two solutions using boundary conditions at $x=0$ ...

1.) ψ is always continuous

$$\psi(x) = \begin{cases} B e^{Kx} & (x \leq 0) \\ B e^{-Kx} & (x \geq 0) \end{cases}$$



$K = ?$

Integrate Schröd. eq. around zero (from $-\epsilon$ to $+\epsilon$ with $\epsilon \to 0$)

$$-\frac{\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} \frac{d^2\psi}{dx^2}\, dx + \int_{-\epsilon}^{\epsilon} V(x)\, \psi(x)\, dx = \underbrace{E \int_{-\epsilon}^{\epsilon} \psi(x)\, dx}_{=0}$$

$$\Downarrow$$

$$\underbrace{\left.\frac{d\psi}{dx}\right|_{+\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon}} = \frac{2m}{\hbar^2} \int_{-\epsilon}^{\epsilon} (-\alpha)\, \delta(x)\, \psi(x)\, dx$$

$$\lim_{\epsilon \to 0} \Downarrow$$

$$\Delta \frac{d\psi}{dx} = -\frac{2m\alpha}{\hbar^2} \psi(0)$$

$(x > 0)$ $\quad \frac{d\psi}{dx} = -BK e^{-Kx} \longrightarrow \left.\frac{d\psi}{dx}\right|_{+} = -BK$

$(x < 0)$ $\quad \frac{d\psi}{dx} = BK e^{Kx} \longrightarrow \left.\frac{d\psi}{dx}\right|_{-} = +BK$ $\qquad$ and $\underline{\psi(0) = B}$

$$\Rightarrow \cancel{-}2\cancel{B}K = \cancel{-}\frac{2m\alpha}{\hbar^2}\cancel{B} \quad \Rightarrow \boxed{K = \frac{m\alpha}{\hbar^2}}$$

$$E = -\frac{\hbar^2 K^2}{2m} = \boxed{-\frac{m\alpha^2}{2\hbar^2}}$$ allowed energy

Normalize ψ

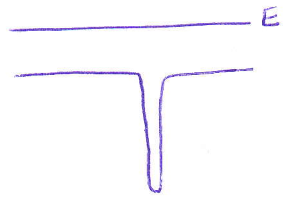
$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2|B|^2 \int_0^{\infty} e^{-2Kx}\, dx =$$

$$= 2|B|^2 \left. \frac{e^{-2Kx}}{-2K} \right|_0^{\infty} = \frac{2|B|^2}{2K} = 1$$

$$B = \sqrt{K} = \frac{\sqrt{m\alpha}}{\hbar}$$

$$\boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha|x|}{\hbar^2}} \quad ; \quad E = \frac{-m\alpha^2}{2\hbar^2}}$$

Scattering States $\quad E > 0$



$x < 0 \quad (E > 0)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$x > 0 \quad (E > 0)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\boxed{k^2 = \frac{2mE}{\hbar^2}} \rightarrow \boxed{E = \frac{\hbar^2 k^2}{2m}}$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi(x) = Fe^{ikx} + Ge^{-ikx}$$

continuity of $\psi(x)$ at $x=0$

$$\boxed{A + B = F + G}$$

can't use continuity of $\frac{d\psi}{dx}$ at $x=0$, because at this point V is infinite

so integrate Schröd. eq. from $-\epsilon$ to ϵ with $\epsilon \to 0$

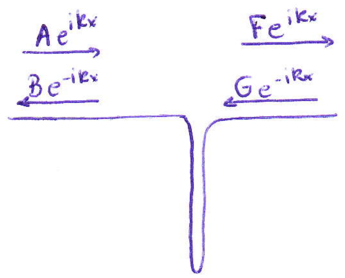
$$-\frac{\hbar^2}{2m}\left(\frac{d\psi}{dx}\Big|_{\epsilon+} - \frac{d\psi}{dx}\Big|_{\epsilon-}\right) - \alpha\,\psi(0) = E\,\underbrace{(\psi(+\epsilon) - \psi(-\epsilon))}_{0}$$

$$ik(F-G) - ik(A-B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

$$(F-G) = \underbrace{\frac{2m\alpha}{\hbar^2 k}}_{\beta} i\,(A+B) + (A-B) \longrightarrow \boxed{F - G = A(1+2i\beta) - B(1-2i\beta)}$$

(!) 2 equations and 5 unknowns

physically



A: amplitude of a wave coming from the left $(-\infty)$

B: " " returing to $-\infty$

G: " " coming from the right $(+\infty)$

F: " " returning to $+\infty$

particles are fired from one direction

Assume from the left $\Rightarrow$
scattering from the left $\Big\}$ $\boxed{G=0}$

A is the amplitude of the incident wave

B is the amplitude of the reflected wave

F is the amplitude of the transmitted wave

$$\left\{\begin{array}{l} A+B=F \\ F=A(1+2i\beta)-B(1-2i\beta) \end{array}\right\} \Rightarrow B=\frac{i\beta}{1-i\beta}A \quad , \quad F=\frac{1}{1-i\beta}A \qquad *$$

Reflection coefficient

$R=\frac{|B|^2}{|A|^2}$

$\boxed{R=\frac{\beta^2}{1+\beta^2}}$

prob that incident particle is reflected back

Transmission coefficient

$T=\frac{|F|^2}{|A|^2}$

$\boxed{T=\frac{1}{1+\beta^2}}$

prob. that incident particle is transmitted

$\boxed{R+T=1}$

$$R = \frac{1}{1 + \frac{2\hbar^2 E}{m\alpha^2}} \qquad T = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}} \qquad (**)$$

higher energy ⇒ greater prob. of transmission (larger T)

smaller energy ⇒ greater R

(*)

$A + B = F$

$F = A(1 + 2i\beta) - B(1 - 2i\beta)$

$\Rightarrow \cancel{A} - \cancel{A} - A2i\beta + B + B - B2i\beta = 0$

$2B(1 - i\beta) = 2i\beta A$

$$\boxed{B = \frac{i\beta}{1 - i\beta} A}$$

$$F = \frac{\cancel{(1 - i\beta)}A}{(1 - i\beta)} + \frac{\cancel{i\beta} A}{(1 - i\beta)}$$

$$\boxed{F = \frac{1}{1 - i\beta} A}$$

(**)

$$R = \frac{\beta^2}{1 + \beta^2}$$

$$\beta = \frac{m\alpha}{\hbar^2 k}$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$T = \frac{1}{1 + \beta^2}$$

$$\beta = \frac{m\alpha\cancel{\hbar}}{\hbar^{\cancel{2}}\sqrt{2mE}}$$

$$\beta = \frac{\alpha}{\hbar}\sqrt{\frac{m}{2E}}$$

$$R = \frac{\alpha^2}{\cancel{\hbar^2}} \frac{m}{\cancel{2E}} \frac{\cancel{\hbar^2 2E}}{(\hbar^2 2E + m\alpha^2)}$$

$$\boxed{R = \frac{1}{1 + \frac{2\hbar^2 E}{m\alpha^2}}} \leftarrow R = \frac{1}{(1 + 1/\beta^2)}$$

$$\boxed{T = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}}}$$