Algebraic Method

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

Let's study the operators:

$$a_+ \equiv \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \qquad a_- \equiv \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x)$$

Motivation: trying to factor the Hamiltonian

$$H = \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]$$

for numbers
$u^2+v^2 = (iu+v)(-iu+v)$
but operators ~~do~~ may not commute

$$a_+ a_- = \frac{1}{2\hbar m\omega}(-ip+m\omega x)(+ip+m\omega x) =$$

$$= \frac{1}{2\hbar m\omega}\left[p^2 + (m\omega x)^2 + im\omega\underbrace{(xp-px)}\right]$$

commutator of x and p

$$\boxed{[A,B] \equiv AB - BA}$$

$xp - px = [x,p]$ → test function

$$[x,p]f(x) = x\frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{d}{dx}(xf) = \frac{\hbar}{i}\left(\cancel{x\frac{df}{dx}} - \cancel{x\frac{df}{dx}} - f\right) = \hbar i f$$

↳ $\boxed{[x,p] = i\hbar}$ canonical commutation relation — this result is at the heart of QM

$$a_+ a_- = \frac{1}{\hbar\omega} \underbrace{\frac{1}{2m}\left[p^2 + (m\omega x)^2\right]}_{H} + \frac{i}{2\hbar}\underbrace{[x,p]}_{i\hbar}$$

$$a_+ a_- = \frac{H}{\hbar\omega} - \frac{1}{2}$$

common notation

$$\begin{cases} a_+ = a^\dagger & \text{(a dagger)} \\ a_- = a \end{cases}$$

$$\boxed{H = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)}$$

verify that $\boxed{H = \hbar\omega\left(a_- a_+ - \frac{1}{2}\right)}$ (see book)

Therefore

$$[a_-, a_+] = a_- a_+ - a_+ a_- = \frac{H}{\hbar\omega} + \frac{1}{2} - \frac{H}{\hbar\omega} + \frac{1}{2} = 1$$

$$\boxed{[a_-, a_+] = 1}$$

Let's prove that

given $H\psi = E\psi$

$(a_+\psi)$ and $(a_-\psi)$ are also solutions of the Schröd eq.

with energy $\Downarrow (E + \hbar\omega)$ $\quad \Downarrow (E - \hbar\omega)$

$(a_+\psi)$

$$H(a_+\psi) = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)(a_+\psi)$$

$$= \hbar\omega\left(a_+a_-a_+ + \frac{1}{2}a_+\right)\psi = a_+\hbar\omega\left(a_-a_+ + \frac{1}{2}\right)\psi$$

$$a_-a_+ = a_+a_- + 1$$

$$= a_+\hbar\omega\left(a_+a_- + \frac{1}{2} + 1\right)\psi$$

$$= a_+(H + \hbar\omega)\psi = a_+(E + \hbar\omega)\psi = (E + \hbar\omega)(a_+\psi)$$

$$H(a_-\psi) = \hbar\omega\left(a_-a_+ - \frac{1}{2}\right)(a_-\psi)$$

$$= \hbar\omega\left(a_-a_+a_- - \frac{1}{2}a_-\right)\psi = a_-\hbar\omega\left(a_+a_- - \frac{1}{2}\right)\psi$$

$$a_-a_+ - 1$$

$$= a_-\hbar\omega\left(a_-a_+ - \frac{1}{2} - 1\right)\psi$$

$$= a_-(H - \hbar\omega)\psi = a_-(E - \hbar\omega)\psi = (E - \hbar\omega)(a_-\psi)$$

ladder operators $\left\{ \begin{array}{l} a_+ \rightarrow \text{raising operator} \\ a_- \rightarrow \text{lowering operator} \end{array} \right\}$ generate new solutions

Ground state ψ_0

$$a_- \psi_0 = 0$$

$$\Rightarrow \frac{1}{\sqrt{2\hbar m\omega}} \underbrace{\left(ip + m\omega x \right)}\psi_0 = 0$$

$$\left(\hbar \frac{d}{dx} + m\omega x\right)\psi_0 = 0 \Rightarrow \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} x \psi_0$$

$$\int \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}\int x\,dx \Rightarrow \ln \psi_0 = -\frac{m\omega x^2}{2\hbar} + \text{const}$$

$$\psi_0 = A e^{-\frac{m\omega x^2}{2\hbar}}$$

normalization $\quad |A|^2 \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} dx = 1 \Rightarrow |A|^2 \sqrt{\frac{\pi\hbar}{m\omega}} = 1$

$$A = (m\omega/\pi\hbar)^{1/4}$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}} \longrightarrow H\psi_0 = E_0\psi_0$$

$$\hbar\omega\left(a_+\cancel{a_-} + \frac{1}{2}\right)\psi_0 = E_0\psi_0$$

$$\Rightarrow \boxed{E_0 = \frac{\hbar\omega}{2}}$$

To generate all excited states

$$\boxed{\psi_n(x) = A_n (a_+)^n \psi_0(x)}$$

from normalization

with $\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega}$

energy increases by $\hbar\omega$ at each step

$$\boxed{H\psi_n = E\psi_n} \rightarrow \hbar\omega\left(a_+a_- + \frac{1}{2}\right)\psi_n = \hbar\omega\left(n+\frac{1}{2}\right)\psi_n$$

$$\Downarrow$$

$$\boxed{a_+a_-\psi_n = n\,\psi_n}$$

Normalization

$$a_+\psi_n = c_n\psi_{n+1} \qquad a_-\psi_n = d_n\psi_{n-1}$$

$$\rightarrow \int_{-\infty}^{\infty}\psi_m^*\,(a_\pm\psi_n)\,dx = \int_{-\infty}^{\infty}(a_\mp\psi_m)^*\,\psi_n\,dx$$

a_- is the hermitian conjugate of a_+

a_+ " of a_-

$$= \frac{1}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty}\psi_m^*\left(\mp\hbar\frac{d}{dx} + m\omega x\right)\psi_n\,dx$$

$$\int\psi_m^*\frac{d\psi_n}{dx}\,dx = \cancel{\psi_m^*\psi_n\Big|} - \int\frac{d\psi_m^*}{dx}\psi_n$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty}\left(\pm\hbar\frac{d}{dx} + m\omega x\right)\psi_m^*\,\psi_n\,dx = \int_{-\infty}^{\infty}(a_\mp\psi_m)^*\,\psi_n\,dx$$

$$\rightarrow \int_{-\infty}^{\infty}(a_+\psi_n)^*(a_+\psi_n)\,dx = \int(a_-a_+\psi_n)^*\psi_n\,dx \overset{[a_-,a_+]=1}{=} \int(a_+a_- + 1)\psi_n^*\,\psi_n\,dx$$

$$= (n+1)\int|\psi_n|^2\,dx = (n+1)$$

$$a_+a_-\psi_n = n\,\psi_n$$

$$|c_n|^2\int|\psi_n|^2dx = |c_n|^2$$

$$|c_n|^2 = n+1 \qquad \boxed{c_n = \sqrt{n+1}}$$

$$\int_{-\infty}^{\infty} (a_- \psi_n)^* (a_- \psi_n)\, dx = \int (a_+ a_- \psi_n)^* \psi_n \, dx = n$$

$$\shortparallel$$

$$|d_n|^2$$

$$\boxed{d_n = \sqrt{n}}$$

$$\boxed{a_+ \psi_n = \sqrt{n+1}\, \psi_{n+1}} \longrightarrow \psi_{n+1} = \frac{1}{\sqrt{n+1}} a_+ \psi_n \Rightarrow$$

$$= \frac{1}{\sqrt{(n+1)(n)}} (a_+)^2 \psi_{n-1} = \frac{1}{\sqrt{(n+1)n(n-1)}} (a_+)^3 \psi_{n-2} = \dots$$

$$\boxed{a_- \psi_n = \sqrt{n}\, \psi_{n-1}}$$

$$= \frac{1}{\sqrt{(n+1)!}} (a_+)^{n+1} \psi_0$$

$$\Rightarrow \psi_1 = a_+ \psi_0 , \qquad \psi_2 = \frac{1}{\sqrt{2}} a_+ \psi_1 = \frac{1}{\sqrt{2}} (a_+)^2 \psi_0$$

$$\psi_3 = \frac{1}{\sqrt{3}} a_+ \psi_2 = \frac{1}{\sqrt{3.2}} (a_+)^2 \psi_1 = \frac{1}{\sqrt{3.2.1}} (a_+)^3 \psi_0$$

$$\psi_4 = \frac{1}{\sqrt{4.3.2.1}} (a_+)^4 \psi_0 \quad \dots$$

$$\boxed{\psi_n(x) = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0} \longleftrightarrow \boxed{E_n = \left(n + \tfrac{1}{2}\right) \hbar\omega}$$

Orthonormality $\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}$

$$\int_{-\infty}^{\infty} \psi_m^* (a_+ a_-) \psi_n \, dx = n \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx$$

$$= \int_{-\infty}^{\infty} (a_- \psi_m)^* (a_- \psi_n)\, dx = \int_{-\infty}^{\infty} (a_+ a_- \psi_m)^* \psi_n \, dx = m \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx$$

unless $m = n$, then $\int \psi_m^* \psi_n \, dx = 0$

(Probls. 2.12 and Example 2.5)

Find $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, $\langle T \rangle$, and $\langle V \rangle$ for the n-th stationary state

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(-ip + m\omega x\right) \qquad a_- = \frac{1}{\sqrt{2\hbar m\omega}}\left(ip + m\omega x\right)$$

$$2m\omega x = \sqrt{2\hbar m\omega}\,(a_+ + a_-)$$

$$\boxed{x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)}$$

$$\rightarrow \langle x \rangle = \int_{-\infty}^{\infty} \psi_n^* \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)\,\psi_n\,dx =$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\int \psi_n^*\psi_{n+1}\,dx + \sqrt{n}\int \psi_n^*\psi_{n-1}\,dx\right) = \boxed{0}$$

$$\rightarrow \langle x^2 \rangle = \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty} \psi_n^* \left(\cancel{(a_+)^2} + a_+a_- + a_-a_+ + \cancel{(a_-)^2}\right)\psi_n\,dx =$$

$a_-\psi_n = \sqrt{n}\,\psi_{n-1}$

$a_+\psi_{n-1} = \sqrt{n}\,\psi_n$

$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$

$a_-\psi_{n+1} = \sqrt{n+1}\,\psi_n$

$$= \frac{\hbar}{2m\omega}\left[n\int |\psi_n|^2\,dx + (n+1)\int |\psi_n|^2\,dx\right] =$$

$$= \frac{\hbar}{2m\omega}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}}$$

2.3-14.

$\rightarrow)\ \langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0}$

$$2ip = \sqrt{2\hbar m w}\,(a_- - a_+)$$

$$\boxed{p = i\sqrt{\frac{\hbar m w}{2}}\,(a_+ - a_-)}$$

$\rightarrow)\ \langle p^2\rangle = -\frac{\hbar m w}{2}\int \psi_n^*\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right)\psi_n\,dx$

$a_-\psi_n = \sqrt{n}\,\psi_{n-1}$

$a_+\psi_{n-1} = \sqrt{n}\,\psi_n$

$$= \frac{\hbar m w}{2}(n+n+1) = \boxed{\left(n+\frac{1}{2}\right)\hbar m w}$$

$\rightarrow)\ \langle T\rangle = \frac{\langle p^2\rangle}{2m} = \frac{1}{2}\left(n+\frac{1}{2}\right)\hbar w$

$\rightarrow)\ \langle V\rangle = \frac{1}{2}mw^2\langle x^2\rangle = \frac{1}{2}\left(n+\frac{1}{2}\right)\hbar w$

$\rightarrow)\ \sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\left(n+\frac{1}{2}\right)\frac{\hbar}{mw}}$ $\qquad \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\left(n+\frac{1}{2}\right)\hbar m w}$

$$\boxed{\sigma_x\sigma_p = \left(n+\frac{1}{2}\right)\hbar} \geq \frac{\hbar}{2}$$