# Section 23 . Harmonic Oscillator

•) classical harmonic oscillator


Hooke's law

$$F = -kx = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x \qquad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

solution: (see NOTES)

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

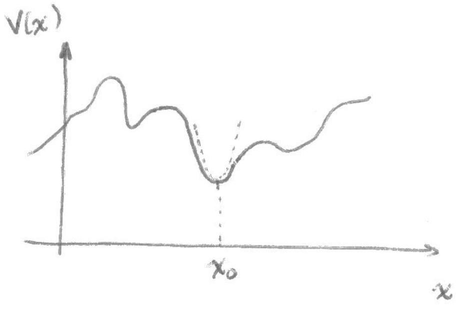
$\omega$ is the angular frequency of oscillations

Potential energy

$$\boxed{V(x) = \frac{1}{2}kx^2} \qquad \text{where } F = -\frac{dV}{dx}$$

is parabolic

•) any potential is approximately parabolic in the neighborhood of a local minima




Taylor series about the minimum $x_0$

$$V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 + \dots$$

$\left\{\begin{array}{l}\text{subtract } V(x_0)\\ V'(x_0)=0 \quad (x_0 \text{ is minimum})\\ \text{drop higher orders}\end{array}\right.$

it describes simple harmonic oscillations about $x_0$ with spring constant $V''(x_0) = k$ $\leftarrow$ $\boxed{V(x) \cong \frac{V''(x_0)}{2}(x-x_0)^2}$

-) Quantum harmonic oscillator

$$\boxed{V(x) = \frac{1}{2} m \omega^2 x^2}$$

time-independent Schrödinger equation:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi}$$

two ways to solve it

-) power series method ← mathematical Physics course

-) algebraic technique – ladder operators

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\boxed{\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}} \longleftrightarrow \boxed{E_n = \left(n+\frac{1}{2}\right)\hbar\omega}$$

$\boxed{n = 0, 1, 2, \ldots}$

quantization of energy

Hermite polynomials

$H_0 = 1$

$H_1 = 2\xi$

$H_2 = 4\xi^2 - 2$

$H_3 = 8\xi^3 - 12\xi$

$\vdots$

$E_0 = \frac{\hbar\omega}{2}$ → ground state energy (or zero-point energy)

(NOT zero)

↳ NO rest even at absolute zero temperature

Stationary states of the harmonic oscillator are orthonormal

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}$$

⇒ given $\Psi(x,0)$, we can find $c_n$ and therefore $\Psi(x,t)$

∘) Use Mathematica ... items a–g

Print plot
$|\Psi_n(x)|^2$ and V

$\boxed{QM}$

① energies are quantized
② odd states – prob. at center is zero
③ higher n ⇒ more bumps ⇒ shorter $\lambda$ ⇒ ⇒ higher momentum ⇒ higher energy

∘) Classical Harmonic oscillator

$$E = T + V$$

classically allowed region

turning points: $T = 0 \Rightarrow E = V \Rightarrow \frac{1}{2} m \omega^2 x^2 = E \Rightarrow \pm x_0 = \pm \sqrt{\frac{2E}{m\omega^2}}$

amplitude of oscillation: $\sqrt{\frac{2E}{m\omega^2}}$

$\boxed{QM}$

④ prob. for finding the particle outside the classically allowed range is NOT zero (tunneling)

↳ for energy $E_n$

↳ $P = 2 \int_{x_0}^{\infty} |\Psi_n|^2 \, dx$

∘) Distribution function for the classical harmonic oscillator

↳ calculated from the time the oscillator spends in ≠ regions

$dt = \frac{dx}{v}$ ⟶ $E = T + V \Rightarrow \frac{1}{2} m v^2 = E - \frac{1}{2} m \omega^2 x^2 \Rightarrow v = \pm \sqrt{\frac{2E}{m} - \omega^2 x^2}$

⇓

$\Downarrow$

$$P_{clas} = \frac{Norm}{v}$$

Normalization: $\displaystyle\int_{-x_0}^{x_0} \frac{Norm}{v}\,dx = 1 \Rightarrow Norm = \frac{1}{\displaystyle\int_{-x_0}^{x_0} \frac{dx}{v}}$

$$\boxed{P_{clas} = \frac{v^{-1}}{\displaystyle\int_{-x_0}^{x_0} v^{-1}\,dx}}$$

$$P_{clas} = \frac{\left(2E/m - \omega^2 x^2\right)^{-1/2}}{\displaystyle\int_{-x_0}^{x_0} \left(2E/m - \omega^2 x^2\right)^{-1/2} dx}$$

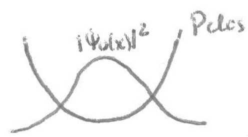
$\hookrightarrow$ amplitude $= \sqrt{\dfrac{2E}{m\omega^2}}$

use mathematica... (h-k)

⑤ $P_{clas}$ vs. $|\psi_n(x)|^2$

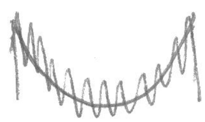
$n=0 \Rightarrow P_{clas}(x) = \dfrac{A_0}{\sqrt{\dfrac{2E_0}{m} - \omega^2 x^2}}$ vs. $|\psi_0|^2$

n=0




$n=100 \Rightarrow P_{clas}(x) = \dfrac{A_{100}}{\sqrt{\dfrac{2E_{100}}{m} - \omega^2 x^2}}$ vs. $|\psi_{100}|^2$

$n \to \infty$



$n \to \infty$ : quantum prob. resembles the classical case

Correspondence Principle:
tendency to approach the classical behavior for high quantum numbers

Good site

http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/hosc.html

*TOPIC for ORAL PRESENTATION: Zero point energy
(Sc. Am. 1985 pp 70-78)

## ■ Quantum Harmonic Oscillator

Let us assume:

m=1

ℏ=1

ω=2

and write the stationary states, the energy levels, and the potential for the harmonic oscillator as:

```
m = 1;
ℏ = 1;
ω = 2;
ξ = Sqrt[m ω / ℏ] x;
psi[nn_] := ((m ω) / (Pi ℏ)) ^ (1 / 4) 1 / Sqrt[2 ^ nn nn !] HermiteH[nn, ξ] Exp[-ξ ^ 2 / 2];
En[nn_] := (nn + 1 / 2) ℏ ω;
V = (1 / 2) m ω ^ 2 x ^ 2;
```

a) Verify that the first four stationary states are indeed normalized.

Verify that the first three stationary states are indeed orthogonal.

b) Plot the first four stationary states psi[nn] (use different colours for each plot).

c) Plot the first four probability densities Abs[psi[nn]]^2 (use different colours for each plot).

d) Plot the potential and the first four energy levels all in the same graph.

e) Add to the plot in item (d) the functions En[nn]+psi[nn] for nn=0,1,2,3.

f) Add to the plot in item (d) the functions En[nn]+Abs[psi[nn]]^2 for nn=0,1,2,3.

g) Blow up the graph from item (f) and print it for analysis.

## ■ Comparison with the Classical Harmonic Oscillator

It is interesting to compare the distribution function for the quantum mechanical harmonic oscillator to the distribution function for a classical harmonic oscillator. For the classical oscillator, the probability distribution function can be calculated from the time the oscillator spends in the various regions. The probability function will be larger in regions where the oscillator spends more time.

h) The classical allowed region for a classical oscillator of energy E extends from -Sqrt[(2E)/(mω^2)] to Sqrt[(2E)/(mω^2)], which are the "classical turning points". The amplitude of the oscillations is then A=Sqrt[(2E)/(mω^2)]. In the ground state of the quantum harmonic oscillator, what is the probability of finding the particle outside the classically allowed region?

i) Evaluate the amplitude of the classical oscillator corresponding to the energy of the first three stationary states of the quantum mechanical harmonic oscillator and note that the amplitude is increasing with energy.

j) What is the velocity v of the classical particle as a function of position (use E=T+V)?
Given that the amount of time the particle spends in an interval *dx* around position *x* is *dx/v*, find the probability function of finding the classical particle on the interval (*-A,A*).
Plot the classical and quantum probability distribution functions for the lowest energy state *n=0*. Why does the classical probability distribution function have sharp maxima at the points of maximum displacement, and the quantum probability does not? Give a qualitative answer.

k) There is a principle of correspondence which states that in the limit of large energy and large objects, the predictions of quantum theory must be the same as predictions from classical dynamics. We know that the macroscopic world of large objects that we can observe is well described by classical dynamics. We say that we reach the 'classical limit' when the energy is large enough that the spacing between the energy levels is insignificant. Show with a plot how the probability distribution of a quantum harmonic oscillator reaches the classical limit as *n* becomes large.