Classical Physics

particle in 1D , $F(x,t)$ } $\Rightarrow$ position at any t

initial conditions } $x(t)$, v, p, $T = \frac{1}{2}mv^2$

$$\underbrace{H}_{\text{Hamiltonian}} = \underbrace{T}_{\text{Kinetic energy}} + \underbrace{V}_{\text{potential energy}}$$

Quantum Mechanics

we have: particle's wavefunction Ψ

$\Psi(x,t)$

Schrödinger eq.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

operator

$$\hat{H}\Psi = \hat{T}\Psi + \hat{V}\Psi$$

$$i\hbar\frac{\partial}{\partial t} \qquad \frac{\hat{p}^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

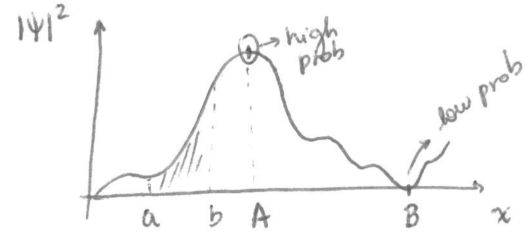
particle $\rightarrow$ localized in space } how to reconcile the two?
$\Psi(x,t) \rightarrow$ spread
↳ represents the state of particle

Born's statistical interpretation of Ψ

$|\Psi(x,t)|^2$ – prob. of finding particle at point x at time t

$\int_a^b |\Psi(x,t)|^2 \, dx$ } prob. of finding particle between a and b at t

prob – area under graph of $|\Psi|^2$



Ψ is complex

QM is intrinsically <u>probabilistic</u>

can't predict with certainty the outcome of experiments

disturbing to { physicists / philosophers } (Einstein: God does not play dice)

<u>measurement problem</u>

after measurement particle is at C

where was right before?

⇒ <u>realist</u> (Einstein)
- particle was at C
- QM is incomplete
- Ψ is not the whole story, additional information (hidden variables)

But Bell's inequality rules out <u>local</u> hidden variables interpretations

<u>non-local</u> remain (Bohm)

⇒ <u>orthodox</u>
- particle was nowhere
- act of measurement forces it to take a stand
- <u>Copenhagen</u> <u>interpretation</u> (Bohr)

⇒ <u>agnostic</u>
- refuses to answer
- can't ask before measuring → metaphysics

<u>Orthodox</u>

after measurement → Ψ collapses ⇒
- repeated measurements
- particle always at C

Parenthesis ...

Probability — discrete variables

$N(14) = 1$ $\longrightarrow \sum_{j=0}^{\infty} N(j) = N = 14$

$N(15) = 1$

$N(16) = 3$ $\longrightarrow P(j) = \frac{N(j)}{N}$ $\left\{ \begin{array}{l} P(15) = 1/14 \\ P(16) = 3/14 \end{array} \right.$

$N(22) = 2$

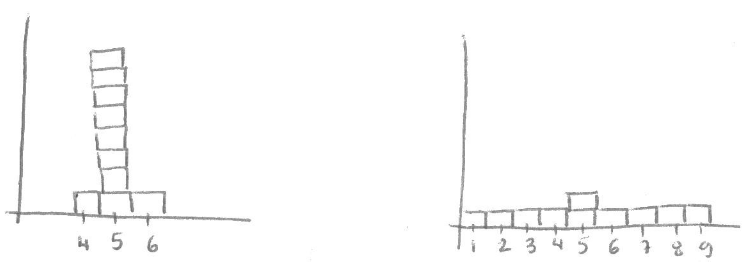
$N(24) = 2$

$N(25) = 5$ $\longrightarrow \sum_{j=0}^{\infty} P(j) = 1$

$$\langle j \rangle = \frac{\sum_{j=0}^{\infty} j\, N(j)}{N} = \sum_{j=0}^{\infty} P(j)\, j$$

$$\langle j^2 \rangle = \sum_{j=0}^{\infty} j^2 P(j)$$

$$\langle f(j) \rangle = \sum_{j=0}^{\infty} f(j)\, P(j)$$



$\left.\right\}$ $N = 10$

$\langle j \rangle = 5$

different distributions

$$\Delta j = j - \langle j \rangle \Rightarrow \langle \Delta j \rangle = \sum (j - \langle j \rangle) P(j) = \underbrace{\sum j P(j)}_{\langle j \rangle} - \langle j \rangle \underbrace{\sum P(j)}_{1} = 0$$

$\langle |\Delta j| \rangle$

chosen $\longrightarrow$ $\langle (\Delta j)^2 \rangle = \sum (j - \langle j \rangle)^2 P(j) = \sum j^2 P(j) - 2\langle j \rangle \underbrace{\sum j P(j)}_{\langle j \rangle} - \langle j \rangle^2 \sum P(j) = \langle j^2 \rangle - \langle j \rangle^2$

$\langle (\Delta j)^4 \rangle$

$\Rightarrow$ <u>VARIANCE</u> $\sigma^2 \equiv \langle (\Delta j)^2 \rangle = \langle j^2 \rangle - \langle j \rangle^2$

$\Rightarrow$ <u>STANDARD DEVIATION</u> $\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$

continuous variables - classical phys.

infinitesimal intervals $\rho(x)$: probability density

$$P_{ab} = \int_a^b \rho(x)\, dx$$ prob. that variable lies between a and b

$\rho(x)\, dx$ prob. to lie between x and $x + dx$

$$\int_{-\infty}^{\infty} \rho(x)\, dx = 1 \qquad \langle x \rangle = \int_{-\infty}^{\infty} \rho(x)\, x\, dx$$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\, \rho(x)\, dx$$

$$\sigma^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$$

back to QM ...

$|\Psi(x,t)|^2$ - prob. density

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = 1 \longrightarrow$$ normalization condition

Ψ normalized at $t=0$, Ψ remains normalized at any t

$$\underbrace{\frac{d}{dt}}_{\text{total derivative}} \int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = 0$$

$$\underbrace{\frac{d}{dt}}_{\substack{\text{total}\\\text{derivative}}}\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx = \int_{-\infty}^{\infty}\underbrace{\frac{\partial}{\partial t}}_{\substack{\text{partial}\\\text{derivative}}}\underbrace{|\Psi(x,t)|^2}_{\Psi^*\Psi}\,dx = \int_{-\infty}^{\infty}\left(\frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t}\right)dx =$$

$$\left\{\begin{array}{l} i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \longrightarrow \frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - i\frac{V}{\hbar}\Psi \\ -i\hbar\frac{\partial\Psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^* \longrightarrow \frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + i\frac{V}{\hbar}\Psi^* \end{array}\right\}$$ REAL potential

$$= \int_{-\infty}^{\infty}\left[\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \cancel{\frac{iV}{\hbar}\Psi^*}\right)\Psi + \Psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{iV}{\hbar}\Psi}\right)\right]dx$$

$$= \int_{-\infty}^{\infty}\frac{i\hbar}{2m}\underbrace{\left(-\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \Psi^*\frac{\partial^2\Psi}{\partial x^2}\right)}_{\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)}dx = \frac{i\hbar}{2m}\left(\underbrace{\Psi^*\frac{\partial\Psi}{\partial x}\Big|_{-\infty}^{\infty}}_{0} - \underbrace{\frac{\partial\Psi^*}{\partial x}\Psi\Big|_{-\infty}^{\infty}}_{0}\right) = \boxed{0}$$

$$\underbrace{\Psi(\pm\infty,t),\ \Psi^*(\pm\infty,t)\to 0}$$

otherwise we couldn't guarantee that $\int_{-\infty}^{\infty}|\Psi|^2dx = 1$

Problem 1.14

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t) \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx$$

probability current $\quad J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$

change in time of the probability in region $a \le x \le b$ is equal to the rate at which probability flows into the region (what enters in a minus what leaves in b)

→ continuity equation (C.E.) $\xrightarrow{\text{associated with}}$ conservation laws

→ conservation of
- mass $\xrightarrow{\text{C.E.}}$ fluid dynamics
- charge $\xrightarrow{\text{C.E.}}$ electromagnetism
- probability $\xrightarrow{\text{C.E.}}$ quantum mechanics

Problem 1.15

unstable particle – decays

τ – lifetime

$P(t)$ – not constant $\Longrightarrow$ $\boxed{V = V_0 - i\Gamma}$ ← REAL and IMAGINARY parts

$V(x,t)$ is real only when there is conservation of probability

Show that

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar} P$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = \int\left(\frac{\partial \Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial \Psi}{\partial t}\right) dx =$$

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + (V_0 - i\Gamma)\Psi \longrightarrow \frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V_0\Psi - \frac{\Gamma}{\hbar}\Psi$$

$$\longrightarrow \frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V_0\Psi^* - \frac{\Gamma}{\hbar}\Psi^*$$

$$= \int_{-\infty}^{\infty}\Bigg(\underbrace{-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \cancel{\frac{i}{\hbar}V_0\Psi^*\Psi} - \frac{\Gamma}{\hbar}\Psi^*\Psi + \frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2}}_{\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)} - \cancel{\frac{i}{\hbar}V_0\Psi^*\Psi} - \frac{\Gamma}{\hbar}\Psi^*\Psi\Bigg)\, dx$$

$$= \frac{i\hbar}{2m}\cancel{\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}} - \frac{2\Gamma}{\hbar}\int|\Psi|^2\, dx = -\frac{2\Gamma}{\hbar}P$$

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}P \Longrightarrow \underline{\overline{P(t) = e^{-2\Gamma t/\hbar}P(0)}} \rightarrow \underline{P(t) = e^{-t/\tau}P(0)}$$

lifetime: $\underline{\overline{\tau = \hbar/2\Gamma}}$

HW

to do	to know	in class
1.4	1.9	1.15
1.5	1.17	
1.7		
1.14		
1.18		

Expectation Values

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, |\Psi(x,t)|^2\, dx$$

ensemble of particles all prepared in the same initial state

$\langle x \rangle \rightarrow$ average of measured results

$\hookrightarrow$ expectation value of x

after measurement, Ψ collapses $\rightarrow$ get always the same result

integration by parts

$$\int_a^b f \frac{dg}{dx}\, dx = fg\Big|_a^b - \int_a^b \frac{df}{dx}\, g\, dx$$

$$\int_a^b \frac{df}{dx}\, g\, dx = fg\Big|_a^b - \int_a^b f \frac{dg}{dx}\, dx$$

$$\frac{d\langle x\rangle}{dt} = \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} |\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} x \left(\frac{\partial \Psi^*}{\partial t}\Psi + \Psi^* \frac{\partial \Psi}{\partial t} \right) dx =$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \longrightarrow \frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{iV}{\hbar}\Psi$$

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{iV}{\hbar}\Psi^*$$

V is REAL

$$= \int_{-\infty}^{\infty} x \left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \cancel{\frac{iV}{\hbar}\Psi^*\Psi} + \frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{iV}{\hbar}\Psi^*\Psi} \right) dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x \frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi \right) dx$$

$$\underset{\substack{\uparrow \\ \text{integration} \\ \text{by} \\ \text{parts}}}{=} \frac{i\hbar}{2m}\left[\cancel{x\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right) dx \right] = -\frac{i\hbar}{2m}\, 2\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\, dx$$

by parts

$$\int_{-\infty}^{\infty} \frac{\partial\Psi^*}{\partial x}\Psi\, dx = \cancel{\Psi^*\Psi\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\, dx$$

$$\langle v\rangle = \frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\, dx$$

$\leftarrow$ velocity of the expectation value of v

NOT the velocity of the particle

expectation value of v

prob. density of velocity – later chapter

$\langle p \rangle = m \langle v \rangle$

$$\boxed{\langle x \rangle = \int \psi^* \underbrace{x}_{\hat{x}} \psi \, dx}$$

sandwich

$$\boxed{\langle p \rangle = \int \psi^* \underbrace{\frac{\hbar}{i} \frac{\partial}{\partial x}}_{\hat{p}} \psi \, dx}$$

$$\hat{T} = \frac{\hat{p}^2}{2m} \Rightarrow \langle T \rangle = -\frac{\hbar^2}{2m} \int \psi^* \frac{\partial^2}{\partial x^2} \psi \, dx$$

$$Q(\hat{x}, \hat{p}) \longrightarrow \langle Q(x,p) \rangle = \int \psi^* \, Q(\hat{x}, \hat{p}) \, \psi \, dx$$

Ehrenfest's theorem

expectation values obey classical laws

class	quant
$v = \frac{dx}{dt}$	$\langle v \rangle = \frac{d\langle x \rangle}{dt}$
$p = m \frac{dx}{dt}$	$\langle p \rangle = m \frac{d\langle x \rangle}{dt}$
$\underbrace{\frac{dp}{dt}}_{F} = -\frac{\partial V}{\partial x}$	$\underbrace{\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle}_{\text{Problem 1.7}}$

Problem 1.7

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^* \frac{\hbar}{i}\frac{\partial\psi}{\partial x}\,dx$$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}\left[\int \frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x}\,dx + \int \psi^*\frac{\partial}{\partial x}\frac{\partial\psi}{\partial t}\,dx\right] =$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \longrightarrow \frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - i\frac{V}{\hbar}\psi$$

$$\frac{\partial\psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + i\frac{V}{\hbar}\psi^*$$

$$= \frac{\hbar}{i}\left[\int_{-\infty}^{\infty}\left(-\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x} + \frac{iV}{\hbar}\psi^*\frac{\partial\psi}{\partial x} + \frac{i\hbar}{2m}\psi^*\frac{\partial^3\psi}{\partial x^3} - \frac{i}{\hbar}\psi^*\frac{\partial}{\partial x}(V\psi)\right)dx\right]$$

$$= \frac{\hbar}{i}\frac{i\hbar}{2m}\int\Bigg(\underbrace{\psi^*\frac{\partial^3\psi}{\partial x^3}}_{\text{by parts}} - \frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}\Bigg)dx + \frac{\hbar}{i}\frac{i}{\hbar}\int\underbrace{\left(V\psi^*\frac{\partial\psi}{\partial x} - \psi^*\frac{\partial(V\psi)}{\partial x}\right)}\,dx = -\int_{-\infty}^{\infty}\psi^*\frac{\partial V}{\partial x}\psi\,dx$$

by parts

$$\cancel{\psi^*\frac{\partial^2\psi}{\partial x^2}\Big|} - \underbrace{\int\frac{\partial\psi^*}{\partial x}\frac{\partial^2\psi}{\partial x^2}}_{\text{by parts}}$$

$$-\cancel{\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\Big|} + \int\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}$$

cancel

$$\cancel{V\psi^*\frac{\partial\psi}{\partial x}} - \psi^*\frac{\partial V}{\partial x}\psi - \cancel{\psi^* V\frac{\partial\psi}{\partial x}}$$

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

Uncertainty Principle.

in any wave phenomenon → duality

- know λ, don't know x

- know x, don't know λ

↓

also appear in QM

in QM

$\lambda \longrightarrow p$

wavelength

de Broglie formula: $\lambda = \frac{h}{p}$, $p = \frac{h}{\lambda} \to p = \frac{\hbar 2\pi}{\lambda}$

$$\Rightarrow \boxed{\sigma_x \sigma_p \geq \hbar/2} \quad \text{Heisenberg uncertainty principle}$$

Problem 1.9

important to know

Gaussian integral $\boxed{\int_{-\infty}^{\infty} e^{-Ax^2} dx = \sqrt{\frac{\pi}{A}}}$ $\Rightarrow \begin{cases} \int_{-\infty}^{\infty} \underbrace{x}_{\text{odd}}\, \underbrace{e^{-Ax^2}}_{\text{even}} dx = 0 \\ \int_{-\infty}^{\infty} x^2 e^{-Ax^2} dx = \frac{1}{2A}\sqrt{\frac{\pi}{A}} \end{cases}$

$$= -\frac{d}{dA}\int_{-\infty}^{\infty} e^{-Ax^2} dx = -\frac{d}{dA}\pi^{1/2}A^{-1/2} = \frac{1}{2}\pi^{1/2}A^{-3/2}$$

$$= \left(\int_{-\infty}^{\infty} e^{-Ax^2} dx \int_{-\infty}^{\infty} e^{-Ay^2} dy\right)^{1/2} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-A(x^2+y^2)} dx\, dy\right)^{1/2} \Rightarrow$$

polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$

$$\Rightarrow \iint dx\, dy\, e^{-A(x^2+y^2)} = \qquad dx\, dy = r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r\, dr\, e^{-Ar^2} \underset{\substack{q = r^2 \\ dq = 2r\,dr}}{=} 2\pi \int_0^{\infty} \frac{dq}{2} e^{-Aq} = \pi \int_0^{\infty} e^{-Aq} dq = \pi \left.\frac{e^{-Aq}}{(-A)}\right|_0^{\infty} = \frac{\pi}{A}$$